# Some Problems of Stabilization and Output Regulation of Nonlinear Systems

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A Thesis Submitted in Partial Fulfilment

of the Requirements for the Degree of

Master of Philosophy

in

Automation and Computer-Aided Engineering



February, 2002

## Abstract

Nonlinear control has been an important subject with both theoretical and practical interest. The techniques of nonlinear control analysis and design can significantly enhance the ability of a control engineer to deal with practical control problems effectively. Since the 1980s, there has been consistent enthusiasm for developing advanced nonlinear control methods to address a number of key issues important in control theory and applications. This thesis aims to make some original contributions to two fundamental nonlinear control problems, namely, *global stabilization* and *output regulation.*

The thesis consists of two parts. In the first part, we consider the global robust stabilization of a class of nonlinear cascaded systems. This problem is challenging when the zero dynamics is not exponentially stable. In the current literature there exist some recursive algorithms for handling this problem utilizing the small gain theorem. However, the procedure cannot guarantee the solvability of the problem due to some stringent conditions imposed on the systems. We will show that, for the cascaded systems with polynomial nonlinearity, it is possible to develop a constructive method to solve this problem. The significance of this result is almost self-evident since the polynomial systems are very frequently encountered in practice.

In the second part of the thesis, we will address the output regulation problem of singular nonlinear systems. Output regulation is one of the central problems in systems and control. Solvability of this problem has been extensively studied for normal nonlinear systems, but is barely studied for singular nonlinear systems. We will present two contributions to the research of this problem as summarized below.

- *Regulation by normal output feedback control:* The existing approach to solving output regulation problem by normal output feedback control relies on the normalizability assumption. By developing a novel approach, we are able to remove this assumption, thus giving a complete solution for this problem.

- *Robust output regulation:* The output regulation problem for singular nonlinear systems has been studied only recently for the ideal case where the mathematical model is exactly known. We will further take into account the model uncertainties, thus offering a more realistic solution. Our work has extended the existing results from the uncertain normal nonlinear systems to the uncertain singular nonlinear systems.

## 摘　要

非線性控制是理論與實踐中的一個重要課題。非線性控制的分析與設計技術能明顯加強控制工程師有效地解決實際控制問題的能力， 並且深化了人們對於具有非線性本質的真實世界的認識。

自二十世紀八十年代以來，現代非線性控制方法的研究展現出持續的進展。一系列控制理論與應用上的關鍵問題獲得深入的探討。本論文旨在對兩個基本的非線性控制問題－－全局鎮定和輸出調節－－做進一步的研究。

本論文分爲兩部分。第一部分探討一類非線性串聯系統的全局魯棒鎮定。當系統的零動態不是指數穩定時，這類問題具有一定的難度。目前，對於解決這類問題已有一些利用小增益理論的遞歸演算法。但是，由於爲保證遞歸過程的有效性而加在系統上的若干嚴格條件，使得問題的可解性不能得到保證。我們將針對一類具有多項式非線性的串聯系統，通過構造性的方法解決全局鎮定問題。由於多項式系統在實踐中普遍存在，因而這個結果具有普適意義。

論文的第二部分闡述奇異非線性系統的輸出調節問題。輸出調節是系統與控制領域中的一個核心問題。對於非奇異的常規系統，這類問題的可解性已經得到了廣泛的研究，對於奇異系統卻是剛剛起步。本論文在這類問題上的探討將集中在如下兩個方面：

- **常規輸出反饋調節**： 現有的通過常規輸出反饋控制解決奇異系統的輸出調節問題的方法依賴於可常規化假設。本文給出了一個新的方法，可去除該假設，從而完滿地解決了該問題。

- **魯棒輸出調節**： 現有的奇異非線性系統的輸出調節問題的解是在理想情況下，亦即系統的數學模型精確可知的情況下得到的，本文將進一步考慮在系統含有不確定參數情況下該問題的解，從而提供了一個更具實用意義的解。

# Acknowledgements

I would like to expressing my heartiful appreciation for the help from those who were heavily involved in the completion of this thesis. First, I would like to thank my supervisor, *Prof. Jie Huang*, for his kind guidance and supervision, especially for his financial support of the first year of my postgraduate study. I have gained a lot of knowledge from him that are applicable for my future career. I would also like to thank *Prof. Yeung Yam*, *Prof. Duan Li* and *Prof. Guanrong Chen* for serving in my thesis committee, and for their valuable suggestions and advices.

My colleagues have played important roles on my progress. I often recall the scenes of our discussion on the problems of our research. Sometimes, we were so excited about certain arguments due to different points of view, but yet the outcomes always turned out to be positive. I will never forget the friendships with *Jin Wang, Dan Wang, Jijun Zhao, Kin Yeung, Waiwing Law, Weiyao Lan, Yichang Liu* and *Cong Qu.*

Finally, I would like to thank my family for their continuous and strong support.

Zhiyong
January, 2002

# Contents

# Chapter 1

# Introduction

Nonlinear control is an important area of systems and control science and engineering. Techniques of nonlinear control analysis and design can significantly enhance the ability of a control engineer to deal with practical control problems effectively. Learning these techniques leads to a sharper understanding of the real world, which is inherently nonlinear. Currently, there is considerable enthusiasm for the research and application of nonlinear control methods in various fields of sciences and engineering.

In general terms, the objective of control design can be stated as follows: *given a physical system to be controlled and the specifications of its desired behavior, construct a feedback control law to make the closed-loop system display the desired behavior.* In accordance with the objective, a number of key issues have to be considered among which *stabilization* and *output regulation* are of fundamental interest.

This chapter provides some background for nonlinear control and the current progresses on the two fundamental issues to be studied in the later chapters. Section 1.1 highlights the motivations for the study of nonlinear control and briefly illustrates some unique and rich behaviors of nonlinear systems. Sections 1.2 and 1.3 review the current research on global stabilization and output regulation problems, respectively, and then explain the motivations of our research. Finally, Section 1.4 gives an overview of the contributions of this thesis.

## 1.1 Nonlinear Control

Linear control is a mature subject with a variety of powerful methods and a long history of successful industrial applications. Therefore, it is not surprised to see so many researchers and designers, from such broad areas as aircraft and spacecraft control, robotics, process control, and biomedical engineering, have a common and great interest in the development and applications of nonlinear control methodologies today. Many reasons can be given to this interest, as is evidenced from the literatures, including improvement of existing control systems, analysis of hard nonlinearities, dealing with model uncertainties, design simplicity, and cost and performance optimality, *etc.* [42]. All these point to a fact, that is, nonlinear control is an important area of systems and control science and engineering.

Physical systems are inherently nonlinear. Thus, all real control systems are nonlinear to a certain extent. Nonlinear control systems can be described by nonlinear differential equations. Nonlinearities can be classified as *inherent* (*natural*) and *intentional* (*artificial*). Inherent nonlinearities are those that naturally come with the system's hardware and motion while intentional nonlinearities are artificially introduced by the designer. Nonlinearities can also be classified in terms of their mathematical properties, as *continuous* and *discontinuous.* If the operating range of a control system is small, and if the involved nonlinearities are smooth, then the control system may be reasonably approximated by a linearized system. Discontinuous nonlinearities cannot be locally approximated by linear functions, and they are also called "hard" nonlinearities. Commonly nonlinear systems show some unique and rich behaviors as compared with linear systems, *e.g.*, multiple isolated equilibrium points, limit cycles, bifurcations and chaos. Other interesting types of behaviors, such as jump resonance, subharmonic generation, and asynchronous quenching, also occur and become important in some nonlinear system studies. Thus, nonlinear systems have considerably richer and more complex behaviors than linear systems in general.

## 1.2 Global Stabilization

In the last decade, methods for global stabilization of nonlinear systems have experienced a vigorous growth. New concepts and techniques, such as input-to-state stability, feedback passivication, small gain theory and backstepping, have greatly increased our ability of designing feedback laws for achieving global stability in various classes of uncertain nonlinear systems.

One of the recent focuses in nonlinear control is global robust stabilization of nonlinear *cascaded* systems. This problem has been approached with Lyapunov's direct method in [15], [25], [28], [29], [30], [33], [36], [37], [39], [41], among others. In particular, an important class of cascaded nonlinear systems was studied in [28], [29], in the presence of both unstructured static uncertainties and stable dynamic uncertainties. Based on the work of [28], a method for robust global stabilization of the systems having a *lower-triangular* structure was presented in [26], where a recursive algorithm for handling this problem was also given. However, the proposed procedure cannot guarantee the solvability of the problem due to some stringent conditions imposed on the systems.

In this thesis, we will study the solution of a class of cascaded systems with polynomial nonlinearities, aiming to develop a constructive method to solve this problem. The significance is almost self-evident since polynomial systems are very frequently encountered in practice. We note that this result is also interesting for the robust nonlinear output regulation problem since this problem can only be solved for polynomial nonlinear systems to date [3], [16], [18].

## 1.3 Output Regulation

Briefly, the output regulation problem is concerned with designing a control law for a plant such that the closed-loop system satisfies two requirements. The first requirement is the (local or nonlocal) closed-loop stability, and the second one is asymptotic tracking and disturbance rejection, *i.e.*, to have the output of the closed-loop system asymptotically track a class of reference inputs in the presence of a class of disturbances. Here, both the reference inputs and disturbances are generated by an autonomous differential equation called *exosystem*. If the uncertainty of the plant is further considered, the problem is called structurally stable output regulation problem (alternatively, robust servomechanism problem).

For the class of *normal* systems, this problem was thoroughly studied for the linear case in the 1970s, in [11], [13] and [14], among others. A salient outcome of these research activities is the *internal model principle*, which enables the conversion of the output regulation problem into an eigenvalue placement problem for an augmented linear system. For nonlinear systems, the same problem was first treated for the special case in which the exogenous signals are constant [10], [14], and [22]. The nonlinear output regulation problem with time varying exogenous signals was first studied in 1990 by Isidori and Byrnes, without considering the parameter uncertainty [27]. Subsequently, the robust version of the same problem was studied in [3], [12], [16], [19], [20], and [31]. These research efforts have led to various methods for synthesizing controllers that can achieve asymptotic tracking and disturbance rejection for an uncertain nonlinear system with local stability.

The fundamental idea of these methods is, similarly to the linear case, the employment of a nonlinear version of the internal model principle.

As for *singular* systems, the study of output regulation has long been limited to linear systems [10], [34], and only very recently a clear-cut solution was obtained by Lin and Dai [34]. More recently, the output regulation problem for singular nonlinear systems has been formulated and solved in part by Huang and Zhang in 1998 [24]. We will further tackle this problem in two aspects in this thesis, as summarized below.

*Regulation by normal output feedback control:* The existing approach for solving the output regulation problem by normal output feedback control relies on the restrictive normalizability assumption. We will develop a new approach that does not need this assumption.

*Robust output regulation:* The output regulation problem for singular nonlinear systems has recently been studied for the ideal case where the mathematical model is exactly known. We will take the model uncertainty into account and develop a robust version of output regulation for singular nonlinear systems.

## 1.4 Contributions of the Thesis

The contributions of this thesis can be summarized as follows:

- A constructive method for solving the robust global stabilization problem for nonlinear cascaded polynomial systems is developed.

- The normalizability assumption as mentioned above is removed, thus a complete solution for output regulation problem of singular nonlinear systems by normal output feedback control is given.

- Some solvability conditions for the robust output regulation problem of singular nonlinear systems are established, thereby extending the existing results for normal systems to singular systems.

# Chapter 2

# Global Robust Stabilization of Cascaded Polynomial Systems

Global robust stabilization of nonlinear cascaded systems is a challenging problem when the zero dynamics is not exponentially stable. Recently there have been some progress in developing a recursive procedure for handling this problem by utilizing the small gain theorem. However, the procedure cannot guarantee the solvability of the problem since it involves verification of some stringent conditions that arise at each step of the recursion. In this chapter, we will show that, for the important class of cascaded *polynomial* systems, the solvability conditions can be satisfied by appropriately implementing the recursive procedure. This result leads to a systematic construction of the control law.

This chapter is organized as follows: Section 2.1 gives an introduction of this problem. Section 2.2 describes the problem and reviews some existing results. In Section 2.3, some basic results are derived. The main algorithm will then be developed in Section 2.4. Finally, an example is given to illustrate our result.

## 2.1 Introduction

Cascaded systems have been studied in the literature in a variety of forms. The problem of robust stabilization of cascaded nonlinear systems has been approached with Lyapunov's direct method in [15], [25], [28], [29], [30], [33], [36], [37], [39], [41], among others. This problem was first studied for the following system in [28]:

$$\begin{cases} \dot{Z}_i = q_i\left(x_1, \cdots, x_i, Z_i\right), \quad 1 \leq i \leq n \\ \dot{x}_i = x_{i+1} + f_i\left(x_1, \cdots, x_i, Z_i\right), \quad 1 \leq i \leq n \end{cases} \tag{2.1}$$

where $u := x_{n+1} \in \Re$ is the input, $(x_1, \cdots, x_n)^T \in \Re^n$ are the measured components, and $(Z_1, \cdots, Z_n)$ are the unmeasured components of the state vector. A solution by using partial state feedback was given in [28] under the assumption that system $\dot{Z}_i = q_i(x_1, \cdots, x_i, Z_i)$, $i = 1, \cdots n$, is input-to-state stable with $(x_1, \cdots, x_i)$ as input.

Based on the work of [28] and [29], the global robust stabilization problem of the systems described by equations having a *lower-triangular* structure of (2.2) was studied in [26], where a recursive design procedure was given. It was shown there that this recursive procedure works if, in addition to some technical hypotheses, at each step of the recursion, a certain subsystem resulting from the previous steps must be ISS and, in particular, the estimate of a certain gain function associated with the small gain theorem must be known.

This chapter focuses on the class of cascaded nonlinear systems described by (2.2), with an additional assumption that the nonlinearities are of polynomial form. It will be shown that the global robust stabilization problem can be solved under two mild assumptions. Moreover, the designed controller can be explicitly constructed.

## 2.2 Preliminaries

We consider a class of cascaded nonlinear systems described as follows:

$$\begin{cases} \dot{z} = f(z, x_1, \mu) \\ \dot{x}_1 = f_1(z, x_1, \mu) + g_1(z, x_1, \mu) x_2 \\ \vdots \\ \dot{x}_r = f_r(z, x_1, \cdots x_r, \mu) + g_r(z, x_1, \cdots, x_r, \mu) u \end{cases} \tag{2.2}$$

where $z \in \Re^m$, $x_i \in \Re$, $i = 1, \cdots, r$, $u \in \Re$, and $\mu \in P \subset \Re^p$ is a vector of unknown parameters with $P$ a prescribed compact set containing the origin of $\Re^p$. Also, the functions $f$, $f_i$, $g_i$, $i = 1, \cdots, r$, are sufficiently smooth satisfying $f(0, \cdots, 0, \mu) = 0$, $f_i(0, \cdots, 0, \mu) = 0, i = 1, \cdots, r$, for all $\mu \in P$. It is assumed that (2.2) satisfies the following two hypotheses.

(H1): For each $\mu$, the upper subsystem, *i.e.*, $\dot{z} = f(z, x_1, \mu)$, is ISS with $z$ as state and $x_1$ as input, and, in particular, a *class* $K_\infty$ function $\kappa(\cdot)$, locally Lipschitz at the origin and independent of $\mu$, is known such that the response $z(\cdot)$ to any bounded $x_1(\cdot)$ satisfies

$$\|z(t)\| \leq \max\{\beta(\|z(0)\|, t), \kappa(\|x_1(\cdot)\|_\infty)\} \tag{2.3}$$

for some *class* $KL$ function $\beta(\cdot, \cdot)$.

(H2): For $i = 1, \cdots, r$, there exist real numbers $b_i > 0$ such that $g_i(z, x_1, \cdots, x_i, \mu) \geq b_i$ for all $x_1, \cdots, x_i, z$ and all $\mu \in P$.

**Remark 2.1:** System (2.2) is a special case of the class of systems studied in [29]. The state feedback global robust stabilization problem for this system has been investigated in [29] and [26]. A recursive procedure has been developed based on the small gain theorem to handle this problem. At each step of the recursion, however, the procedure will lead to a subsystem of the following form:

$$\begin{cases} \dot{z} = \varphi(z, x, \mu) \\ \dot{x} = \phi(z, x, \mu) + \psi(z, x, \mu) u \end{cases} \tag{2.4}$$

in which $(z, x) \in \Re^m \times \Re$, $\varphi(0, 0, \mu) = 0$, $\phi(0, 0, \mu) = 0$ for $\mu \in P \subset \Re^p$. The success of this procedure depends on whether or not, at each step, this subsystem satisfies four conditions described in Lemma 11.4.1 of [26]. For convenience, let us rephrase Lemma 11.4.1 of [26] as follows:

**Lemma 2.2:** Consider system (2.4). Suppose the following:

(i) For each $\mu$, the upper subsystem in (2.4) is ISS and, in particular, a *class* $K_\infty$ function $\kappa(\cdot)$, independent of $\mu$, is known such that the response $z(\cdot)$ to any bounded $x(\cdot)$ satisfies

$$\|z(t)\| \leq \max\{\beta(\|z(0)\|, t), \kappa(\|x(\cdot)\|_\infty)\} \tag{2.5}$$

for some *class* $KL$ function $\beta(\cdot, \cdot)$.

(ii) There exists a real number $b_0 > 0$ such that $\psi(z, x, \mu) \geq b_0$ for all $(z, x) \in \Re^m \times \Re$ and all $\mu \in P$.

(iii) For all $(z, x) \in \Re^m \times \Re$ and all $\mu \in P$,

$$\max\left\{|\phi(z, x, \mu)|, |x| |\psi(z, x, \mu)|^2\right\} \leq \max\{\rho_0(|x|), \rho_1(\|z\|)\}$$

where $\rho_0(\cdot)$ and $\rho_1(\cdot)$ are locally Lipschitzian *class* $K$ functions.

(iv) The function $\rho_1(\kappa(\cdot))$ is locally Lipschitz at the origin.

Then, there exists a smooth function $\alpha(x)$, with $\alpha(0) = 0$, such that, under the control law

$$u = \alpha(x) + v \tag{2.6}$$

the closed-loop system (2.4) and (2.6), viewed as a system with input $v$ and state $(z, x)$, is ISS and, in particular, a *class* $K_\infty$ function $\tilde{\kappa}(\cdot)$, which is locally Lipschitz at the origin and independent of $\mu$, is known such that the response $Z(\cdot) = (z(\cdot), x(\cdot))$ to any bounded $v(\cdot)$ satisfies

$$\|Z(t)\| \leq \max\left\{\tilde{\beta}(\|Z(0)\|, t), \tilde{\kappa}(\|v(\cdot)\|_\infty)\right\}$$

for some *class KL* function $\tilde{\beta}(\cdot, \cdot)$.

**Remark 2.3:** Among the four conditions, the first one can be satisfied under assumption H1, by an appropriate design of the control law of the form (2.6), and the second one is always satisfied under assumption H2. But neither the procedure itself nor assumptions H1 and H2 can guarantee the satisfaction of conditions (iii) and (iv). This is because, at each step, the specified functions $\varphi$, $\phi$ depend not only on the functions $f$, $f_i$, $g_i$ of the original system but also on the function $\alpha$ designed at the previous steps.

In this chapter, we will focus our attention on an important class of systems of the form (2.2), with an additional assumption that the function $f$ is polynomial in $z$, $x_1$, and the functions $f_i$, $g_i$, $i = 1, \cdots, r$, are polynomials in $z, x_1, \cdots, x_i$ with all coefficients depending on $\mu$. Such systems are called *polynomial systems.* We will show that, within the existing framework, for polynomial systems of the form (2.2), under assumptions H1 and H2, conditions (iii) and (iv) of Lemma 2.2 are automatically satisfied. Moreover, if we further assume that the *class* $K_\infty$ function $\kappa(\cdot)$ in assumption H1 is in a polynomial form, then, at each step of the recursion, the function $\alpha(x)$ in (2.6) also takes a polynomial form, and can be constructed explicitly. As a result, the global robust stabilization can always be solved *via* a constructive approach.

## 2.3 Basic Results

In this section, we will establish some preliminary results that will lay the foundation of our further approach to be introduced in Section 2.4.

**Lemma 2.4:** For any function $f(z, x, \mu)$ with $(z, x) \in \Re^m \times \Re$, which is polynomial in $(z, x)$, with the coefficients depending on $\mu$ and satisfies $f(0, 0, \mu) = 0$ for any $\mu \in P$, there exist *class K* polynomial functions (thus locally Lipschitz at the origin) $\rho_0(\cdot), \rho_1(\cdot)$ such that

$$|f(z, x, \mu)| \leq \max\{\rho_0(|x|), \rho_1(\|z\|)\} \tag{2.7}$$

*Proof:* Assume $z = \{z_1, \cdots, z_m\}^T$. Because $f(z,x,\mu)$ is a polynomial in $z, x$, there exists a positive integer $k$ such that $f(z,x,\mu)$ can be written in the following form:

$$f(z,x,\mu) = \sum_{i=1}^{k} \sum_{n_{i,1}+\cdots+n_{i,m}+n_{i,x}=i} \alpha_{(n_{i,1},\cdots,n_{i,m},n_{i,x})}(\mu) z_1^{n_{i,1}} \cdots z_m^{n_{i,m}} x^{n_{i,x}}$$

where $\alpha_{(\cdot)}(\mu)$ may depend on $\mu$. Since $\mu \in P$ with $P$ being a compact set, there exist non-negative integers $\alpha_{(\cdot)}$ such that

$$|\alpha_{(n_{i,1},\cdots,n_{i,m},n_{i,x})}(\mu)| \leq \alpha_{(n_{i,1},\cdots,n_{i,m},n_{i,x})}$$

Therefore,

$$\begin{aligned} |f(z,x,\mu)| &\leq \sum_{i=1}^{k} \sum_{n_{i,1}+\cdots+n_{i,m}+n_{i,x}=i} \alpha_{(n_{i,1},\cdots,n_{i,m},n_{i,x})} |z_1|^{n_{i,1}} \cdots |z_m|^{n_{i,m}} |x|^{n_{i,x}} \\ &\leq \sum_{i=1}^{k} \sum_{n_{i,1}+\cdots+n_{i,m}+n_{i,x}=i} \alpha_{(n_{i,1},\cdots,n_{i,m},n_{i,x})} \|z\|^{(n_{i,1}+\cdots+n_{i,m})} |x|^{n_{i,x}} \end{aligned}$$

Now let $\rho_0(s) = \rho_1(s) = \sum_{i=1}^{k} \bar{\alpha}_i s^i$, where

$$\bar{\alpha}_i = \sum_{n_{i,1}+\cdots+n_{i,m}+n_{i,x}=i} \alpha_{(n_{i,1},\cdots,n_{i,m},n_{i,x})}$$

Then

$$\begin{aligned} \rho_0(|x|) &= \sum_{i=1}^{k} \bar{\alpha}_i |x|^i \\ &= \sum_{i=1}^{k} ( \sum_{n_{i,1}+\cdots+n_{i,m}+n_{i,x}=i} \alpha_{(n_{i,1},\cdots,n_{i,m},ni,x)}) |x|^i \\ &= \sum_{i=1}^{k} \sum_{n_{i,1}+\cdots+n_{i,m}+n_{i,x}=i} \alpha_{(n_{i,1},\cdots,n_{i,m},ni,x)} |x|^{(n_{i,1}+\cdots+n_{i,m})} |x|^{n_{i,x}} \\ &\geq |f(z,x,\mu)|, \text{ when } |x| \geq \|z\| \end{aligned}$$

Similarly, we have $|f(z,x,\mu)| \leq \rho_1(\|z\|)$ when $|x| \leq \|z\|$. □

**Lemma 2.5:** Consider system (2.4) with $\phi(z,x,\mu)$ and $\psi(z,x,\mu)$ being polynomials in $z$ and $x$. Suppose the following:

(i′) For each $\mu$, the upper subsystem in (2.4) is ISS and, in particular, a *class* $K_\infty$ function $\kappa(\cdot)$, locally Lipschitz at the origin and independent of $\mu$, is known such that the response $z(\cdot)$ to any bounded $x(\cdot)$ satisfies

$$\|z(t)\| \leq \max\{\beta(\|z(0)\|, t), \kappa(\|x(\cdot)\|_\infty)\}$$

for some *class* $KL$ function $\beta(\cdot,\cdot)$.

(ii′) Same as condition (ii) of Lemma 2.2.

Then, there exists a sufficiently smooth function $\alpha(x)$, with $\alpha(0) = 0$, such that, under the control law

$$u = \alpha(x) + v \tag{2.8}$$

the closed-loop system (2.4) and (2.8), viewed as a system with input $v$ and state $(z, x)$, is ISS and, in particular, a *class* $K_\infty$ function $\tilde{\kappa}(\cdot)$, which is locally Lipschitz at the origin and independent of $\mu$, is known such that the response $Z(\cdot) = (z(\cdot), x(\cdot))$ to any bounded $v(\cdot)$ satisfies

$$\|Z(t)\| \leq \max\left\{\tilde{\beta}(\|Z(0)\|, t), \tilde{\kappa}(\|v(\cdot)\|_\infty)\right\} \tag{2.9}$$

for some *class* $KL$ function $\tilde{\beta}(\cdot,\cdot)$.

*Proof:* By Lemma 2.4, there are *class* $K$ polynomial functions $\rho_0^1(\cdot), \rho_1^1(\cdot)$, $\rho_0^2(\cdot), \rho_1^2(\cdot)$ such that

$$|\phi(z, x, \mu)| \leq \max\{\rho_0^1(|x|), \rho_1^1(\|z\|)\}, \;\; |x\psi^2(z, x, \mu)| \leq \max\{\rho_0^2(|x|), \rho_1^2(\|z\|)\}$$

Let $\rho_0(\cdot) = \rho_0^1(\cdot) + \rho_0^2(\cdot)$, and $\rho_1(\cdot) = \rho_1^1(\cdot) + \rho_1^2(\cdot)$. Then clearly, system (2.4) satisfies the conditions (iii) and (iv) of Lemma 2.2. As a result, system (2.4) satisfies all conditions of Lemma 2.2. Therefore, there exists a sufficiently smooth function $\alpha(x)$, with $\alpha(0) = 0$, such that, under the control law of the form (2.6), the closed-loop system (2.4) and (2.6), viewed as a system with input $v$ and state $(z, x)$, is ISS and, in particular, a *class* $K_\infty$ function $\tilde{\kappa}(\cdot)$, independent of $\mu$, is known such that the response $Z(\cdot) = (z(\cdot), x(\cdot))$ to any bounded $v(\cdot)$ satisfies (2.9) for some *class* $KL$ function $\tilde{\beta}(\cdot,\cdot)$.

It remains to show $\tilde{\kappa}(\cdot)$ is locally Lipschitz at the origin. To this end, recall from the proof of Lemma 11.4.1 [26] that

$$\tilde{\kappa}(r) = \max\{2\kappa \circ \kappa_v(r), 2\kappa_v(r)\} \tag{2.10}$$

where $\kappa_v(r) = dr$ for some positive integer $d = \frac{1}{\sqrt{2b_0 - \epsilon}}$ with $2b_0 > \epsilon > 0$. Next, using the fact that for any non-negative integers $a, b, c, d$,

$$|\max(a, b) - \max(c, d)| \leq |a - c| + |b - d|$$

yields that, for any $x, y \in [0, \delta]$, where $\delta \in R$ is sufficiently small,

$$\begin{aligned} |\tilde{\kappa}(x) - \tilde{\kappa}(y)| &\leq |2\kappa \circ \kappa_v(x) - 2\kappa \circ \kappa_v(y)| + |2\kappa_v(x) - 2\kappa_v(y)| \\ &\leq 2L|\kappa_v(x) - \kappa_v(y)| + 2|\kappa_v(x) - \kappa_v(y)| \\ &\leq (2L + 2)d|x - y| \end{aligned}$$

where $L$ is the Lipschitz constant of function $\kappa(\cdot)$ in $[0, \delta]$. So $\tilde{\kappa}(\cdot)$ is locally Lipschitz at the origin. $\square$

**Remark 2.6:** In Lemma 2.5, if the *class* $K_\infty$ function $\kappa(\cdot)$ is assumed to be polynomial function, then the functions $\alpha$ in (2.8) and $\tilde{\kappa}(\cdot)$ in (2.9) can also be polynomials. Moreover, from the proof of Lemma 11.4.1 of [26], the function $\alpha(x)$ takes the form $\alpha(x) = -x - \hat{\alpha}(x)$, where $\hat{\alpha}(x)$ is a smooth and strictly increasing odd function satisfying

$$\hat{\alpha}(|x|) \geq \frac{3}{2b_0} \max\{\rho_0(|x|), \rho_1(\kappa(|2x|))\} \tag{2.11}$$

where $b_0$, and $\kappa(\cdot)$ are those defined in Lemma 2.5, and $\rho_0(\cdot)$, and $\rho_1(\cdot)$ are those given in the proof of Lemma 2.5. Since $\rho_0(\cdot), \rho_1(\cdot)$ are polynomial functions, it is always possible to choose $\hat{\alpha}$, hence $\alpha$, to be a polynomial function provided the function $\kappa(\cdot)$ is also in polynomial form. In fact, it suffices to let $\hat{\alpha}(x) = ax + bx^p$ with $a$ and $b$ being sufficiently large real numbers, and $p$ a sufficiently large integer. Furthermore, it suffices to choose

$$\tilde{\kappa}(r) = 2\kappa \circ \kappa_v(r) + 2\kappa_v(r)$$

to satisfy (2.10), where clearly $\tilde{\kappa}(r)$ is polynomial.

## 2.4 The Algorithm

Lemma 2.5 together with Remark 2.6 shows that it is always possible to give an explicit expression for controller (2.6). These three numbers, $a, b$ and $p$, may depend on the size of the compact set $P$. We now proceed to further present a constructive algorithm to obtain an explicit controller to solve the problem.

**Theorem 2.7:** Consider system (2.2), under assumptions H1 and H2, and further assume that the *class* $K_\infty$ function $\kappa(\cdot)$ in assumption H1 is in polynomial form. Then the global robust stabilization problem is solvable.

*Proof:* A constructive proof is given. Throughout this proof, denote $x_{r+1} := u$.

*Step 1:* Define a subsystem of (2.2) as follows:

$$\begin{cases} \dot{z} = f(z, x_1, \mu) \\ \dot{x}_1 = f_1(z, x_1, \mu) + g_1(z, x_1, \mu) x_2 \end{cases} \tag{2.12}$$

Under H1 and H2, by Lemma 2.5 and Remark 2.6 there exists a polynomial function $\alpha_1(x_1)$ such that the coordinate transformation $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2 - \alpha_1(\tilde{x}_1)$ converts system

(2.2) into the following:

$$\begin{cases} \dot{Z}_1 = F_1\left(Z_1, \tilde{x}_2, \mu\right) \\ \dot{\tilde{x}}_2 = \tilde{f}_2\left(Z_1, \tilde{x}_2, \mu\right) + \tilde{g}_2\left(Z_1, \tilde{x}_2, \mu\right) x_3 \\ \dot{x}_i = f_i^1\left(Z_1, \tilde{x}_2, x_3 \cdots, x_i, \mu\right) \\ +g_i^1(Z_1, \tilde{x}_2, x_3, \cdots, x_i, \mu) x_{i+1}, i = 3, \cdots, r \end{cases} \tag{2.13}$$

where

$$Z_1 = \begin{pmatrix} z \\ \tilde{x}_1 \end{pmatrix}, \quad F_1\left(Z_1, \tilde{x}_2, \mu\right) = \begin{bmatrix} f(z, x_1, \mu) \\ f_1\left(z, x_1, \mu\right) + g_1\left(z, x_1, \mu\right)\left(\alpha_1(x_1) + \tilde{x}_2\right) \end{bmatrix}$$

$$\tilde{f}_2\left(Z_1, \tilde{x}_2, \mu\right) = f_2\left(z, x_1, x_2, \mu\right) - \Delta\alpha_1(\tilde{x}_1)\left(f_1\left(z, x_1, \mu\right) + g_1\left(z, x_1, \mu\right)\left(\alpha_1(x_1) + \tilde{x}_2\right)\right)$$
$$\tilde{g}_2\left(Z_1, \tilde{x}_2, \mu\right) = g_2(z, x_1, x_2, \mu)$$
$$f_i^1\left(Z_1, \tilde{x}_2, x_3 \cdots, x_i, \mu,\right) = f_i\left(z, x_1 \cdots, x_i, \mu,\right)$$
$$g_i^1(Z_1, \tilde{x}_2, x_3 \cdots, x_i, \mu) = g_i(z, x_1, \cdots, x_i, \mu)$$

with $\Delta\alpha(x) = \frac{d\alpha(x)}{dx}$. Moreover, the subsystem governing $Z_1$ is ISS with state $Z_1$ and input $\tilde{x}_2$, and, in particular, a *class* $K_\infty$ function $\kappa_1(\cdot)$ in polynomial form, locally Lipschitz at the origin, and independent of $\mu$, is known such that the response $Z_1(\cdot)$ to any bounded $\tilde{x}_2(\cdot)$ satisfies

$$\|Z_1(t)\| \leq \max\left\{\beta_1\left(\|Z_1(0)\|, t\right), \kappa_1(\|\tilde{x}_2(\cdot)\|_\infty)\right\}$$

for some *class* $KL$ function $\beta_1(\cdot,\cdot)$. That is, the hypothesis H1 associated with the subsystem governing $Z_1$ holds. Also, the functions $\tilde{f}_2, \tilde{g}_2$ and $f_i^1, g_i^1$, for $i = 3, \cdots, r$, are all polynomials.

*Step* $j$, $j = 2, \cdots, r$ :

Assume at the end of the $(j-1)\,th$ step, that we obtain a system of the form

$$\begin{cases} \dot{Z}_{j-1} = F_{j-1}\left(Z_{j-1}, \tilde{x}_j, \mu\right) \\ \dot{\tilde{x}}_j = \tilde{f}_j\left(Z_{j-1}, \tilde{x}_j, \mu\right) + \tilde{g}_j\left(Z_{j-1}, \tilde{x}_j, \mu\right) x_{j+1} \\ \dot{x}_i = f_i^{j-1}\left(Z_{j-1}, \tilde{x}_j, x_{j+1} \cdots, x_i, \mu\right) \\ +g_i^{j-1}(Z_{j-1}, \tilde{x}_j, x_{j+1}, \cdots, x_i, \mu) x_{i+1}, i = j+1, \cdots, r \end{cases} \tag{2.14}$$

where

$$Z_{j-1} = col\left(z, \tilde{x}_1, \cdots, \tilde{x}_{j-1}\right)$$

and the functions $\tilde{f}_j, \tilde{g}_j$ and $f_i^{j-1}, g_i^{j-1}$, for $i = j+1, \cdots, r$, are all polynomials. Furthermore, hypothesis H1 associated with the subsystem governing $Z_{j-1}$ holds for a known *class* $K_\infty$ function $\kappa_{j-1}(\cdot)$, which is polynomial, thus locally Lipschitz at the origin, and independent of $\mu$, for some *class* $KL$ function $\beta_{j-1}(\cdot,\cdot)$.

Comparing system (2.14) and (2.2) shows that these two systems have the same structure under the same assumptions. Thus, using the same analysis as that in step 1 on system (2.14), it is easy to conclude that there exists a sufficiently smooth function $\alpha_j(\tilde{x}_j)$ such that the coordinate transformation $\tilde{x}_{j+1} = \bar{x}_{j+1} - \alpha_j(\tilde{x}_j)$ converts system (2.14) into the following:

$$\begin{cases} \dot{Z}_j = F_j(Z_j, \tilde{x}_{j+1}, \mu) \\ \dot{\tilde{x}}_{j+1} = \tilde{f}_{j+1}(Z_j, \tilde{x}_{j+1}, \mu) + \tilde{g}_{j+1}(Z_j, \tilde{x}_{j+1}, \mu)\, x_{j+2} \\ \dot{x}_i = f_i^j(Z_j, \tilde{x}_{j+1}, x_{j+2}\cdots, x_i, \mu) \\ +g_i^j(Z_j, \tilde{x}_{j+1}, x_{j+2}, \cdots, x_i, \mu) x_{i+1}\ , i = j+2, \cdots, r \end{cases}$$

where

$$Z_j = col\,(z, \tilde{x}_1, \cdots, \tilde{x}_j)$$

and the functions $\tilde{f}_{j+1}, \tilde{g}_{j+1}$ and $f_i^j, g_i^j$, for $i = j+2, \cdots, r$, are all polynomials. Furthermore, hypothesis H1 associated with the subsystem governing $Z_j$ holds for a known *class* $K_\infty$ function $\kappa_j(\cdot)$, which is polynomial, thus locally Lipschitz at the origin, and independent of $\mu$, for some *class* $KL$ function $\beta_j(\cdot,\cdot)$.

*Step r+1:* At the end of step $r$, one has obtained a system

$$\dot{Z}_r = F_r(Z_r, \tilde{x}_{r+1}, \mu) \tag{2.15}$$

where

$$Z_r = col\,(z, \tilde{x}_1, \cdots, \tilde{x}_r)$$

is ISS with state $Z_r$ and input $\tilde{x}_{r+1}$. Therefore, letting $\tilde{x}_{r+1} = 0$ shows that the origin of the system (2.15) is globally asymptotically stable. Or, noting that $\tilde{x}_{r+1} = x_{r+1} - \alpha_r(\tilde{x}_r)$, the original system (2.2) is globally asymptotically stable under the control $u = \alpha_r(\tilde{x}_r)$. Thus, the overall controller expressed in the original coordinate is given by

$$\begin{cases} u = \alpha_r(\tilde{x}_r) \\ \tilde{x}_i = x_i - \alpha_{i-1}(\tilde{x}_{i-1}), i = r, \cdots, 2 \\ \tilde{x}_1 = x_1 \end{cases} \tag{2.16}$$

□

**Remark 2.8:** For some $i \in \{1, \cdots, r\}$, if there exists a real number $b_i < 0$ such that $g_i(x_1, \cdots, x_i, z, \mu) \leq b_i$ for all $x_1, \cdots, x_i, z$ and all $\mu \in P$, Theorem 2.7 still holds. And, if the upper subsystem of (2.2) is not in polynomial form, that is, the function $f$ is not polynomial, Theorem 2.7 still holds.

**Remark 2.9:** It should be noted that the global stabilization problem for the lower triangular systems of form (2.2) has also been studied under some other assumptions, in [33] and [35]. In particular, in [33], under the assumption that the zero dynamics of system (2.2) is globally asymptotically stable (GAS) and locally exponentially stable (LES), a full state feedback control law can be constructed by a backstepping procedure. Nevertheless, our result here can handle some cases that do not satisfy the LES assumption, as can be seen from the example given in the next section.

**Remark 2.10:** It can be verified that it suffices to assume hypotheses H1 and H2 in order to guarantee the solvability of the global robust stabilization for the polynomial systems (2.2). The additional assumption that the *class* $K_\infty$ function $\kappa(\cdot)$ in hypothesis H1 is a polynomial can further guarantee an explicit construction of a family of control laws in polynomial form.

**Remark 2.11:** Note that, even for polynomial systems with the ISS property, the *class* $K_\infty$ function $\kappa(\cdot)$ mentioned in H1 may not be polynomial, and may not even be locally Lipschitzian. For example, consider the system

$$\dot{x}_1 = -x_1^3 + x_1 x_2$$

which is ISS with state $x_1$ and input $x_2$. An estimate of the form (2.3) holds with $\kappa(r) = \frac{\sqrt{r}}{\sqrt{1-\epsilon}}$ for any $0 < \epsilon < 1$. But $\kappa(\cdot)$ is neither polynomial nor locally Lipschitzian.

## 2.5 An Example

Consider the following lower-triangular system:

$$\begin{cases} \dot{z} = -z^3 + \mu_1 z^2 x_1 \\ \dot{x}_1 = \frac{1}{4}x_1^2 + \frac{1}{6}x_1 z + \mu_1 z + 2x_2 \\ \dot{x}_2 = x_1^2 z + 2x_1 z^2 - \mu_2 x_2 + 5(\mu_3^2 + 1)u \end{cases} \tag{2.17}$$

Let us design a state-feedback controller to globally stabilize this system in the presence of three uncertain parameters $\mu_1, \mu_2, \mu_3$.

For this purpose, we need to check whether or not the subsystem $\dot{z} = -z^3 + \mu_1 z^2 x_1$ of (2.17) satisfies H1. This is indeed the case with the function $\kappa(|x_1|) = |x_1|$, although the system with $x_1 = 0$ is not locally exponentially stable. Thus, by Theorem 2.7, system (2.17) is globally stabilizable with a control law in polynomial form. To explicitly give our control law, we assume $\mu_1, \mu_2, \mu_3 \in [-1, 1]$.

At the first step, consider the subsystem out of (2.17) as follows:

$$\begin{cases} \dot{z} = -z^3 + \mu_1 z^2 x_1 \\ \dot{x}_1 = \frac{1}{4}x_1^2 + \frac{1}{6}x_1 z + \mu_1 z + 2x_2 \end{cases} \tag{2.18}$$

with $x_2$ being viewed as the input and $z, x_1$ as the states.

Let

$$\rho_0(|x_1|) = \frac{1}{3}|x_1|^2 + 4|x_1|, \rho_1(|z|) = \frac{1}{3}|z|^2 + |z|$$

Then

$$\max\left\{\left|\frac{1}{4}x_1^2 + \frac{1}{6}x_1 z + \mu_1 z\right|, 4\,|x_1|\right\} \le \max\{\rho_0(|x_1|), \rho_1(|z|)\}$$

It is easy to verify that $\hat{\alpha}_1(x_1) = x_1^3 + 4x_1$ satisfies

$$\hat{\alpha}_1(|x_1|) \ge |x_1|^2 + 3|x_1| \ge \frac{3}{2 \times 2}\max\{\rho_0(|x_1|), \rho_1(\kappa(|2x_1|))\}$$

So, under controller $x_2 = \alpha_1(x_1) + \tilde{x}_2$, with $\alpha_1(x_1) = -x_1 - \hat{\alpha}_1(x_1) = -x_1^3 - 5x_1$, system (2.18) is ISS with $\tilde{x}_2$ as input, where the estimate $\kappa_1$ can be calculated as $\kappa_1(r) = r$.

This controller defines the coordinate transformation $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2 + x_1^3 + 5x_1$, which puts the original system (2.17) into the following form:

$$\begin{cases} \dot{z} = -z^3 + \mu_1 z^2 \tilde{x}_1 \\ \dot{\tilde{x}}_1 = \frac{1}{4}\tilde{x}_1^2 + \frac{1}{6}\tilde{x}_1 z + \mu_1 z - 2\tilde{x}_1^3 - 10\tilde{x}_1 + 2\tilde{x}_2 \\ \dot{\tilde{x}}_2 = \tilde{x}_1^2 z + 2\tilde{x}_1 z^2 - \mu_2(\tilde{x}_2 - \tilde{x}_1^3 - 5\tilde{x}_1) + (3x_1^2 + 5)\dot{\tilde{x}}_1 + 5(\mu_3^2 + 1)u \end{cases} \tag{2.19}$$

Denote $Z_1 = (z, \tilde{x}_1)^T$. Then

$$\begin{aligned} & |\tilde{x}_1^2 z + 2\tilde{x}_1 z^2 - \mu_2(\tilde{x}_2 - \tilde{x}_1^3 - 5\tilde{x}_1) + (3x_1^2 + 5)\dot{\tilde{x}}_1| \\ \le\ & 6\|Z_1\|^5 + \frac{5}{4}\|Z_1\|^4 + 47\|Z_1\|^3 + \frac{25}{12}\|Z_1\|^2 + 60\|Z_1\| + 6\|Z_1\|^2|\tilde{x}_2| + 11|\tilde{x}_2| \end{aligned}$$

Thus,

$$\begin{aligned} & \max\{|\tilde{x}_1^2 z + 2\tilde{x}_1 z^2 - \mu_2(\tilde{x}_2 - \tilde{x}_1^3 - 5\tilde{x}_1) + (3x_1^2 + 5)\dot{\tilde{x}}_1|, 25|\tilde{x}_2|(\mu^2 + 1)^2\} \\ \le\ & \max\{\rho_0(|\tilde{x}_2|), \rho_1(\|Z_1\|)\} \end{aligned}$$

where

$$\rho_0(|\tilde{x}_2|) = 6|\tilde{x}_2|^5 + \frac{5}{4}|\tilde{x}_2|^4 + 53|\tilde{x}_2|^3 + \frac{25}{12}|\tilde{x}_2|^2 + 100|\tilde{x}_2|$$

$$\rho_1(\|Z_1\|) = 6\|Z_1\|^5 + \frac{5}{4}\|Z_1\|^4 + 53\|Z_1\|^3 + \frac{25}{12}\|Z_1\|^2 + 100\|Z_1\|$$

It is easy to verify that $\hat{\alpha}_2(\tilde{x}_2) = \frac{3}{2}(2\tilde{x}_2)^5 + 9(2\tilde{x}_2)^3 + 25(2\tilde{x}_2) - \tilde{x}_2$ satisfies

$$\hat{\alpha}_1(|\tilde{x}_2|) \geq \frac{3}{2 \times 10} \max\{\rho_0(|\tilde{x}_2|), \rho_1(\kappa(|2\tilde{x}_2|))\}$$

So, under controller $u = \alpha_2(\tilde{x}_2)$, with $\alpha_2(\tilde{x}_2) = -x_2 - \hat{\alpha}_2(\tilde{x}_2) = -\frac{3}{2}(2\tilde{x}_2)^5 - 9(2\tilde{x}_2)^3 - 25(2\tilde{x}_2)$, system (2.17) can be globally stabilized.

The overall controller for solving the global robust stabilization problem for system (2.17) is thus given by

$$\begin{cases} u = -\frac{3}{2}(2\tilde{x}_2)^5 - 9(2\tilde{x}_2)^3 - 25(2\tilde{x}_2) \\ \tilde{x}_2 = x_2 + x_1^3 + 5x_1 \end{cases}$$

Figure 2.1 shows the trajectories of the states for the case where

$$z\,(0) = x_1\,(0) = x_2\,(0) = 100, \mu_1 = -0.4, \mu_2 = 0.8, \mu_3 = 0.3$$

## 2.6 Concluding Remarks

This chapter shows that, for polynomial systems of the form (2.2), the global robust stabilization problem can be solved under hypotheses H1 and H2. Furthermore, a state feedback control law in polynomial form can be explicitly constructed if it is further assumed that the *class* $K_\infty$ function $\kappa(\cdot)$ in hypothesis H1 is a polynomial.

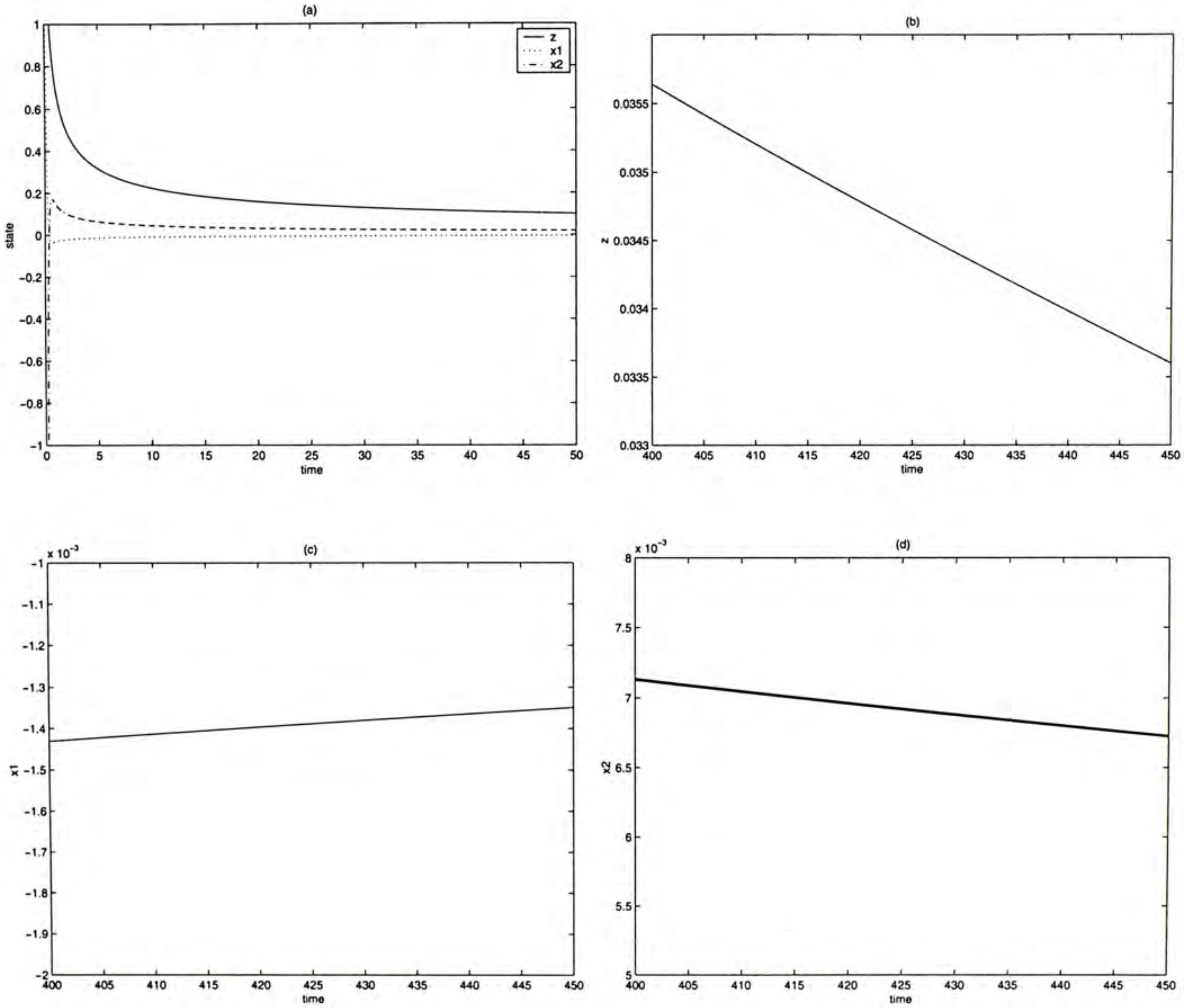


Figure 2.1: (a) shows the trajectories of all states, and (b), (c), (d) respectively show the detailed convergence behaviors of states $z, x_1$ and $x_2$ in small scale.

# Chapter 3

# Output Regulation of Singular Nonlinear Systems by Normal Output Feedback

*Singular* systems are dynamical systems subject to algebraic constraints, and arise in many engineering disciplines. Output regulation of singular nonlinear systems *via* normal output feedback controls has been a challenging problem. Existing approaches for solving this problem employed techniques similar to those used for linear singular systems. Results from these approaches either rely on a normalizability assumption or are limited to systems with special structures. This chapter gives a complete solution to this problem by employing a novel approach that is even interesting for linear systems.

This chapter is organized as follows. Section 3.1 gives an introduction to the problem and Section 3.2 describes the problem and prepares some preliminaries. In Section 3.3, the main result is established.

## 3.1 Introduction

Singular systems arise in many engineering areas such as electrical networks, power systems, aerospace engineering and chemical processing. Over the past two decades, there has been extensive study on singular systems encompassing such issues as solvability, controllability and observability, pole assignment and elimination of impulsive behavior, LQG control, output regulation, input-output decoupling, *etc.* [5], [8], [10], [32] . Such efforts not only have extended a substantial portion of the research results on normal systems to this more general class of dynamic systems, but also have led to many practical applications involving economics, power systems, robot control, and so on.

This and the next chapters will consider the output regulation problem for a class of singular nonlinear systems to be described in Section 3.2. Roughly, by output regulation, we mean to design control laws for a system so that the output of the closed-loop system is able to asymptotically track a class of reference inputs and reject a class of disturbances, both generated by an *exosystem*.

This chapter addresses an open problem in the area of output regulation of singular systems, *i.e*, the output regulation problem of singular nonlinear systems *via* normal output feedback control. The same problem was first studied by Huang and Zhang in [24], where it was shown that the problem can be solved by a normal output feedback control if it can be solved by a singular output feedback control and the system satisfies a normalizability assumption. This assumption is somewhat restrictive, since it cannot be satisfied by many systems. Moreover, for the class of singular *linear* systems, similar hypothesis can be removed by a normalizability decomposition technique [10]. This simple fact has motivated the recent work by Wang and Huang [46], in which they have removed this assumption for a class of nonlinear systems to be specified in Section 3.2. The results obtained in [24] and [46] are limited due to the fact that the technique used there is an extension of the similar one developed for linear singular systems [34]. There are some inherent obstacles in carrying over these techniques to the nonlinear setting. In this chapter, we look into this problem from a different perspective, and develop a novel approach to tackle it. We have indeed succeeded in removing the normalizability assumption for general singular nonlinear systems, thus leading to a complete solution to the problem. Roughly, our major result can be summarized as follows: The output regulation problem of singular nonlinear systems can be solved by a normal output feedback control if and only if the problem can be solved by a singular output feedback control. This result is interesting not only because it bridges the gap between the linear and the nonlinear cases, but also because a normal controller is of lower order, and is much easier to implement in practice.

## 3.2 Preliminaries

Consider a singular plant described by

$$\begin{cases} \mathbf{E}\dot{x}(t) = f(x(t), u(t), v(t)), \quad x(0) = x_0 \\ y(t) = h(x(t), v(t)), \qquad t \geq 0 \end{cases} \tag{3.1}$$

and an exosystem described by

$$\dot{v}(t) = a(v(t)), \quad v(0) = v_0 \tag{3.2}$$

where $x(t) \in \Re^n$ is the plant state, $u(t) \in \Re^m$ the plant input, $y(t) \in \Re^p$ the plant output representing the tracking error, $v(t) \in \Re^q$ the exogenous signal representing the disturbance and/or the reference input, and $\mathbf{E} \in \Re^{n \times n}$ a singular constant matrix, with $rank\,(\mathbf{E}) = n_E < n$. It is noted that if $n_E = n$, the plant is *normal*.

The class of output feedback control laws to be used is described by

$$\begin{cases} u(t) = k(z(t), y(t)) \\ \mathbf{E}_z \dot{z}(t) = g(z(t), y(t)) \end{cases} \tag{3.3}$$

where $z(t)$ is the compensator state vector of dimension $n_z$, and $\mathbf{E}_z \in \Re^{n_z \times n_z}$ is a constant matrix. Equation (3.3) is said to be a *normal* controller if $\mathbf{E}_z$ is an identity matrix.

The closed-loop system, composed of plant (3.1), exosystem (3.2) and control law (3.3) can be put into the form of

$$\begin{cases} \mathbf{E}_c \dot{x}_c(t) = f_c(x_c(t), v(t)), x_c(0) = x_{c0} \\ \dot{v} = a(v(t)), v(0) = v_0 \\ y(t) = h_c(x_c(t), v(t)) \end{cases} \tag{3.4}$$

where

$$\begin{cases} x_c = \begin{bmatrix} x \\ z \end{bmatrix}, \mathbf{E}_c = \begin{bmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{E}_z \end{bmatrix}, h_c(x_c, v) = h(x, v), \\ f_c(x_c, v) = \begin{bmatrix} f(x, k(z, h(x, v)), v) \\ g(z, h(x, v)) \end{bmatrix} \end{cases} \tag{3.5}$$

Throughout this chapter, it is assumed that all the functions involved in this setup are sufficiently smooth and defined globally on appropriate Euclidean spaces, and $a(0) = 0$, $f(0, 0, 0) = 0$, and $h(0, 0) = 0$. Our results will be stated locally in terms of $V$, with $V$ being an open neighborhood of the origin in $\Re^q$. In the sequel, $V$ is implicitly permitted to be made small so as to accommodate subsequent local arguments.

Linear approximations of the plant and the exosystem at the origin will be frequently used. Therefore, the following notation is given:

$$A = \frac{\partial f}{\partial x}|_{x=u=v=0}, B = \frac{\partial f}{\partial u}|_{x=u=v=0},$$

$$E = \frac{\partial f}{\partial v}|_{x=u=v=0}, C = \frac{\partial h}{\partial x}|_{x=v=0},$$

$$F = \frac{\partial h}{\partial v}|_{x=v=0}, A_1 = \frac{\partial a(v)}{\partial v}|_{v=0}, A_c = \frac{\partial f_c}{\partial x_c}|_{x_c=v=0}$$

Using this notation, the linearization of (3.1) and (3.2) is described by

$$\begin{cases} \mathbf{E}\dot{x}(t) = Ax(t) + Bu(t) + Ev(t) \\ y(t) = Cx(t) + Fv(t) \\ \dot{v}(t) = A_1 v(t) \end{cases}$$

*The output regulation problem:* Find a control law such that the closed-loop system (3.4) has the following two properties:

(P1) The linearization at $x_c = 0$ of

$$\mathbf{E}_c \dot{x}_c(t) = f_c(x_c(t), 0)$$

is strongly stable in the sense that

$$\deg(\det(\lambda \mathbf{E}_c - A_c)) = rank(\mathbf{E}_c)$$

and that $\sigma(\mathbf{E}_c, A_c) \in C^-$, where

$$\sigma(\mathbf{E}_c, A_c) = \{\lambda | \det(\lambda \mathbf{E}_c - A_c) = 0\}$$

(P2) The trajectories starting from all sufficiently small initial state $(x_{c0}, v_0)$ satisfy

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} h_c(x_c(t), v(t)) = 0$$

Also, let us list the following standard hypotheses:

(H1): $v = 0$ is a stable equilibrium of the exosystem (3.2), and there exists a neighborhood $V$ in the origin of $\Re^q$ with the property that each initial condition $v_0 \in V$ is stable in the sense of Poisson, that is, all the eigenvalues of $(\partial a/\partial v)(0)$ are simple and have zero real parts.

(H2): $(\mathbf{E}, A, B)$ is strongly stabilizable, *i.e.*, there exists a matrix $K \in \Re^{m\times n}$ such that $(\mathbf{E}, A + BK)$ is strongly stable.

(H3): $\left( \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} C & F \end{bmatrix} \right)$ is strongly detectable, *i.e.* there exists a matrix $G_1 \in \Re^{n\times p}, G_2 \in \Re^{q\times p}$ such that $\left( \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & A_1 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} \right)$ is strongly stable.

(H4): $(\mathbf{E}, B)$ is normalizable, that is, there exists $L \in \Re^{m\times n}$ such that $\mathbf{E} + BL$ is nonsingular.

**Remark 3.1:** The theorem given in [24] shows the solvability of the output regulation problem by a singular controller. It states that, under Hypotheses H1, H2 and H3, the output regulation problem *via* a *singular* output feedback controller is solvable if and only if there exist sufficiently smooth functions $\mathbf{x}(v)$ and $\mathbf{u}(v)$, with $\mathbf{x}(0) = 0$ and $\mathbf{u}(0) = 0$, both defined in a neighborhood $V$ of the origin of $\Re^q$, such that

$$\begin{cases} \mathbf{E}\frac{\partial \mathbf{x}(v)}{\partial v}a(v) = f(\mathbf{x}(v), \mathbf{u}(v), v) \\ h(\mathbf{x}(v), v) = 0 \end{cases} \tag{3.6}$$

When $\mathbf{E}$ is an identity matrix, Hypotheses (H2) and (H3) are reduced to exactly the same ones assumed by Isidori and Byrnes in [27] for the output regulation problem of the *normal* systems; Hypothesis (H4) is automatically satisfied; and (3.6) becomes the so-called regulator equations discovered also by Isidori and Byrnes. Thus, this result can be viewed as an extension of what was obtained in [27] for normal systems to singular systems.

Precisely, the singular output feedback controller is in the form of

$$\begin{aligned} \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} f(z_1, \mathbf{u}(z_2) + H(z_1 - \mathbf{x}(z_2)), z_2) - G_1[h(z_1, z_2) - y] \\ a(z_2) - G_2[h(z_1, z_2) - y] \end{bmatrix} \\ u &= \mathbf{u}(z_2) + H(z_1 - \mathbf{x}(z_2)) \end{aligned}$$

where $H$, $G_1$ and $G_2$ exist for assumptions H2 and H3 such that

$$(\mathbf{E}, A + BH) \text{ and } \left( \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A - G_1C & E - G_1F \\ -G_2C & A_1 - G_2F \end{bmatrix} \right) \text{ are strongly stable.}$$

The output feedback controller constructed here is also singular due to the singularity assumption on $\mathbf{E}$. It is known that singular controllers are sensitive to the variations of initial conditions, and to structured uncertainties. Moreover, it is less easy to realize singular controllers physically. Thus, it is desirable to synthesize normal controllers to solve the problem, if ever possible. The following remark shows that it is indeed possible under an additional hypothesis (H4).

**Remark 3.2:** It was shown in Theorem 4 of [24] that, under Hypotheses (H1) to (H4), the output regulation problem is solvable by a normal output feedback controller if and only if this problem is solvable by a singular output feedback controller.

To continue the above discussion, the normal output feedback control law can be constructed as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \eta(z,y) \\ a(z_2) - G_2[h(z_1,z_2) - y] \end{bmatrix}$$
$$u = \gamma(z, \eta(z,y))$$

As a matter of fact, Hypotheses H2 and H4 together imply the existence of two matrices, $H$ and $L$, such that

$$\mathbf{E} + BL \text{ is nonsingular and } (\mathbf{E} + BL, A + BH) \text{ is strongly stable.}$$

Also, Hypothesis (H3) implies the existence of two matrices, $G_1$ and $G_2$, such that

$$\left( \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A - G_1C & E - G_1F \\ -G_2C & A_1 - G_2F \end{bmatrix} \right) \text{ is strongly stable.}$$

Let

$$\gamma(z, \dot{z}_1) = \mathbf{u}(z_2) + H[z_1 - \mathbf{x}(z_2)] - L\left[\dot{z}_1 - \frac{\partial \mathbf{x}}{\partial z_2} a(z_2)\right]$$

and

$$g_1(z, \dot{z}_1, y) = \mathbf{E}\dot{z}_1 - f(z_1, \gamma(z, \dot{z}_1), z_2) + G_1[h(z_1, z_2) - y]$$

Then it can be verified that

$$\frac{\partial g_1}{\partial \dot{z}_1}(0,0,0) = E + BL$$

Since $E + BL$ is nonsingular, by the Implicit Function Theorem, there exists a sufficiently smooth and locally defined function $\eta(z,y)$, satisfying $\eta(0,0) = 0$, such that

$$g_1(z, \eta(z,y), y) = 0 \tag{3.7}$$

**Remark 3.3:** The output regulation problem can be solved for a more general class of normal nonlinear systems, in which the error output is allowed to depend on the input $u$, as studied by Huang and Rugh in [23],

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), v(t)), & x(0) = x_0 \\ e(t) = h(x(t), u(t), v(t)), & t \geq 0 \end{cases} \tag{3.8}$$

This fact is instrumental for the success of our approach (also see Remark 3.11). For the later convenience, we summarize the solvability conditions for the output regulation problem for normal systems (3.8) as follows ([27], [23]):

Under Hypotheses H1 to H3, the output regulation problem for systems (3.8) *via* an *output feedback controller* is solvable if and only if there exist sufficiently smooth functions $\mathbf{x}(v)$ and $\mathbf{u}(v)$, with $\mathbf{x}(0) = 0$ and $\mathbf{u}(0) = 0$, both defined in a neighborhood $V$ of the origin of $\Re^q$, such that

$$\begin{cases} \frac{\partial \mathbf{x}(v)}{\partial v} a(v) = f(\mathbf{x}(v), \mathbf{u}(v), v) \\ h(\mathbf{x}(v), \mathbf{u}(v), v) = 0 \end{cases}$$

**Remark 3.4:** As mentioned in the Introduction, Hypothesis H4 is somewhat restrictive. Wang and Huang [46] had removed the Hypothesis H4 for a class of nonlinear systems, in which $f(x, u, v)$ is linear in $u$ with a constant input gain, *i.e.*, $f(x, u, v) = Ax + Bu + Ev + \Phi(x, v)$, where $\Phi(\cdot, \cdot)$ is a sufficiently smooth function vanishing at the origin together with its first-order derivatives. The assumption that $f(x, u, v)$ is linear in $u$ with constant input gain is needed so that the normalizability decomposition technique for linear systems can be carried over to nonlinear systems.

## 3.3 Main Result

In this section, we will present a completely different approach to design a normal output feedback controller to solve the output regulation problem for singular nonlinear systems. This approach does not need hypotheses H4, and applies to a general class of nonlinear systems as discussed in last section.

As a first step, we will establish a result that replaces the normalizability assumption by a condition that can be removed later through an output feedback precompensator.

**Lemma 3.5:** Under hypotheses H1 to H3, and the following additional condition:

$$\deg\left(\det\left(\lambda \mathbf{E} - A\right)\right) = n_E \tag{3.9}$$

the output regulation problem of system (3.1) and (3.2) *via* a *normal output feedback controller* is solvable if and only if there exist sufficiently smooth functions $\mathbf{x}(v)$ and $\mathbf{u}(v)$, with $\mathbf{x}(0) = 0$ and $\mathbf{u}(0) = 0$, both defined in a neighborhood $V$ of the origin of $\Re^q$, satisfying equation (3.6).

*Proof:* The necessity follows trivially from the previous result [24]. The proof of sufficiency can be divided into three steps. In the first step, we apply the standard coordinate transformation to the system. This transformation will in turn lead to a well-defined reduced-order normal system. In the second step, we show that the output regulation problem for the normal system obtained in step 1 is solvable. Finally, we show that this normal output controller also solves the output regulation problem for the original system.

*Step 1:* Consider the original system (3.1). There exist two nonsingular matrices $T_1, T_2 \in \Re^{n\times n}$ such that

$$T_1 \mathbf{E} T_2 = \bar{\mathbf{E}} = \begin{bmatrix} I_{n_E} & 0 \\ 0 & 0 \end{bmatrix}$$

Let

$$T_1 A T_2 = \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix},$$

$$T_1 B = \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix},$$

$$T_1 E = \bar{E} = \begin{bmatrix} \bar{E}_1 \\ \bar{E}_2 \end{bmatrix},$$

$$C T_2 = \bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix},$$

$$T_2^{-1} x = \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

where $\bar{A}_{11} \in \Re^{n_E\times n_E}, \bar{B}_1 \in \Re^{n_E\times m}, \bar{E}_1 \in \Re^{n_E\times q}, \bar{C}_1 \in \Re^{p\times n_E}, \bar{x}_1 \in \Re^{n_E}$ and all other matrices have proper dimensions.

This coordinate transformation leads to the following singular system:

$$\dot{\bar{x}}_1 = \bar{f}_1(\bar{x}, u, v) = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + \bar{B}_1 u + \bar{E}_1 v + o(\bar{x}, u, v) \tag{3.10}$$

$$0 = \bar{f}_2(\bar{x}, u, v) = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{x}_2 + \bar{B}_2 u + \bar{E}_2 v + o(\bar{x}, u, v) \tag{3.11}$$

$$y(t) = \bar{h}(\bar{x}, v) = \bar{C}\bar{x} + Fv + o(\bar{x}, v) \tag{3.12}$$

where the notation $o(x)$ denotes higher-order terms in $x$, and

$$\begin{pmatrix} \bar{f}_1(\bar{x}, u, v) \\ \bar{f}_2(\bar{x}, u, v) \end{pmatrix} = T_1 f(x, u, v)$$
$$\bar{h}(\bar{x}, v) = h(x, v)$$

The system described by (3.10) to (3.12) possesses two properties that $(\bar{\mathbf{E}}, \bar{A}, \bar{B})$ is strongly stabilizable, and that

$$\left( \begin{bmatrix} \bar{\mathbf{E}} & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} \bar{A} & \bar{E}_1 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} \bar{C} & F \end{bmatrix} \right)$$

is strongly detectable, because

$$\begin{aligned}
&\det(\lambda\mathbf{E}-(A+BK)) = \det\left(T_1^{-1}\left(\lambda T_1\mathbf{E}T_2-(T_1AT_2+T_1BKT_2)\right)T_2^{-1}\right)\\
=\;&\det\left(T_1^{-1}\right)\times\det\left(T_2^{-1}\right)\times\det\left(\lambda\bar{\mathbf{E}}-\left(\bar{A}+\bar{B}K'\right)\right)\ \textit{with}\ K'=KT_2
\end{aligned}$$

and

$$\begin{aligned}
&\det\left(\lambda\begin{bmatrix}\mathbf{E} & 0\\ 0 & I_q\end{bmatrix}-\left(\begin{bmatrix}A & E\\ 0 & A_1\end{bmatrix}-\begin{bmatrix}G_1\\ G_2\end{bmatrix}\begin{bmatrix}C & F\end{bmatrix}\right)\right)\\
=\;&\det\left(\begin{bmatrix}T_1^{-1} & 0\\ 0 & I_q\end{bmatrix}\right)\times\det\left(\begin{bmatrix}T_2^{-1} & 0\\ 0 & I_q\end{bmatrix}\right)\\
&\times\det\left(\lambda\begin{bmatrix}\bar{\mathbf{E}} & 0\\ 0 & I_q\end{bmatrix}-\left(\begin{bmatrix}\bar{A} & \bar{E}_1\\ 0 & A_1\end{bmatrix}-\begin{bmatrix}G_1'\\ G_2\end{bmatrix}\begin{bmatrix}\bar{C} & F\end{bmatrix}\right)\right)\\
&\textit{with}\ G_1'=T_2G_1
\end{aligned}$$

Moreover, condition (3.9) implies that $\bar{A}_{22}$ is non-singular because

$$\begin{aligned}
&\deg(\det(\lambda\mathbf{E}-A)) = \deg(\det(T_1(\lambda\mathbf{E}-A)T_2))\\
=\;&\deg\left(\begin{bmatrix}\lambda I_{n_E}-\bar{A}_{11} & -\bar{A}_{12}\\ -\bar{A}_{21} & -\bar{A}_{22}\end{bmatrix}\right)=n_E
\end{aligned}$$

Now, by the Implicit Function Theorem, there exists a unique sufficiently smooth function, $\bar{x}_2=\alpha(\bar{x}_1,u,v)$, satisfying $\alpha(0,0,0)=0$, and

$$0=\bar{A}_{21}\bar{x}_1+\bar{A}_{22}\alpha(\bar{x}_1,u,v)+\bar{B}_2u+\bar{E}_2v+o(\bar{x}_1,\alpha(\bar{x}_1,u,v),u,v)$$

It is easy to show that

$$\begin{aligned}
&\bar{x}_2=\alpha(\bar{x}_1,u,v) \qquad (3.13)\\
=\;&-\bar{A}_{22}^{-1}\left(\bar{A}_{21}\bar{x}_1+\bar{B}_2u+\bar{E}_2v\right)+o(\bar{x}_1,u,v)
\end{aligned}$$

Substituting (3.13) into (3.10) and (3.12) gives a reduced-order normal system:

$$\left\{\begin{array}{l}
\dot{\bar{x}}_1=f_r(\bar{x}_1,u,v)=\bar{f}_1\left(\begin{bmatrix}\bar{x}_1\\ \alpha(\bar{x}_1,u,v)\end{bmatrix},u,v\right)\\
=A_r\bar{x}_1+B_ru+E_rv+o(\bar{x}_1,u,v)\\
y=h_r(\bar{x}_1,u,v)=\bar{h}\left(\begin{bmatrix}\bar{x}_1\\ \alpha(\bar{x}_1,u,v)\end{bmatrix},v\right)\\
=C_r\bar{x}_1+Ru+F_rv+o(\bar{x}_1,u,v)
\end{array}\right. \qquad (3.14)$$

where

$$
\begin{aligned}
A_r &= \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21} \\
B_r &= \bar{B}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2 \\
C_r &= \bar{C}_1 - \bar{C}_2\bar{A}_{22}^{-1}\bar{A}_{21} \\
E_r &= \bar{E}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{E}_2 \\
F_r &= F - \bar{C}_2\bar{A}_{22}^{-1}\bar{E}_2 \\
R &= -\bar{C}_2\bar{A}_{22}^{-1}\bar{B}_2
\end{aligned}
$$

*Step 2:* System (3.14) is a normal system. We will show in this step that the output regulation problem for this system is solvable. By Remark 3.3, it suffices to verify that the regulator equation associated with (3.14) is solvable and Hypotheses H2 and H3 are satisfied with $\mathbf{E}$ being an identity matrix.

In fact, let $\bar{\mathbf{x}}(v)=T_2^{-1}\mathbf{x}(v)$ and denote $\bar{\mathbf{x}}(v) = \begin{bmatrix} \bar{\mathbf{x}}_1(v) \\ \bar{\mathbf{x}}_2(v) \end{bmatrix}$. Then $\mathbf{u}(v)$ and $\bar{\mathbf{x}}(v)$ satisfy

$$
\begin{cases}
\frac{\partial \bar{\mathbf{x}}_1(v)}{\partial v} a(v) = \bar{f}_1(\bar{\mathbf{x}}(v), \mathbf{u}(v), v) \\
0 = \bar{f}_2(\bar{\mathbf{x}}(v), \mathbf{u}(v), v)
\end{cases} \tag{3.15}
$$

$$
\bar{h}(\bar{\mathbf{x}}(v), v) = 0 \tag{3.16}
$$

Also it is clear that

$$
\bar{\mathbf{x}}_2(v) = \alpha(\bar{\mathbf{x}}_1(v), \mathbf{u}(v), v) \tag{3.17}
$$

Substituting (3.17) into (3.15) and (3.16) gives

$$
\begin{cases}
\frac{\partial \bar{\mathbf{x}}_1(v)}{\partial v} a(v) = f_r(\bar{\mathbf{x}}_1(v), \mathbf{u}(v), v) \\
h_r\left(\bar{\mathbf{x}}_1(v), \mathbf{u}(v), v\right) = 0
\end{cases} \tag{3.18}
$$

Thus, the pair $(\bar{\mathbf{x}}_1(v), \mathbf{u}(v))$ is the solution of the regulator equation associated with system (3.14).

It remains to show that (3.14) satisfies H2 and H3. To this end, first recall that $(\bar{\mathbf{E}}, \bar{A}, \bar{B})$ is strongly stabilizable, *i.e.*, there exists a matrix $K \in \Re^{m\times n}$ such that $(\bar{\mathbf{E}}, \bar{A} + \bar{B}K)$ is strongly stable.

Denote $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$. Then

$$
\begin{aligned}
& \det\left(\lambda\bar{\mathbf{E}} - (\bar{A} + \bar{B}K)\right) \\
= \; & \det\left(-\bar{A}_{22} - \bar{B}_2K_2\right) \times \det\left\{\lambda I_{n_E} - (\bar{A}_{11} + \bar{B}_1K_1)\right. \\
& \left. + (\bar{A}_{12} + \bar{B}_1K_2)\left(\bar{A}_{22} + \bar{B}_2K_2\right)^{-1}\left(\bar{A}_{21} + \bar{B}_2K_1\right)\right\}
\end{aligned}
$$

Noting that $\bar{A}_{22}$ is nonsingular and using the fact that

$$\left(\bar{A}_{22}+\bar{B}_2K_2\right)^{-1}=\bar{A}_{22}^{-1}-\bar{A}_{22}^{-1}\bar{B}_2\left(K_2\bar{A}_{22}^{-1}\bar{B}_2+I\right)^{-1}K_2\bar{A}_{22}^{-1}$$

give

$$\begin{aligned}&\left(\bar{A}_{11}+\bar{B}_1K_1\right)-\left(\bar{A}_{12}+\bar{B}_1K_2\right)\left(\bar{A}_{22}+\bar{B}_2K_2\right)^{-1}\left(\bar{A}_{21}+\bar{B}_2K_1\right)\\ =&\left(\bar{A}_{11}-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)+\left(\bar{B}_1-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2\right)K'=A_r+B_rK'\end{aligned}$$

where

$$K'=K_1-\left(K_2\bar{A}_{22}^{-1}\bar{B}_2+I\right)^{-1}K_2\bar{A}_{22}^{-1}\left(\bar{A}_{21}+\bar{B}_2K_1\right)$$

Therefore,

$$\det\left(\lambda\bar{\mathbf{E}}-\left(\bar{A}+\bar{B}K\right)\right)=\det\left(-\bar{A}_{22}-\bar{B}_2K_2\right)\times\det\left(\lambda I_{n_E}-\left(A_r+B_rK'\right)\right)$$

which concludes that the pair $(A_r, B_r)$ is stabilizable.

Next, recall that

$$\left(\begin{bmatrix}\bar{\mathbf{E}} & 0\\ 0 & I_q\end{bmatrix},\begin{bmatrix}\bar{A} & \bar{E}\\ 0 & A_1\end{bmatrix},\begin{bmatrix}\bar{C} & F\end{bmatrix}\right)$$

is strongly detectable. Let

$$\begin{aligned}M_1&=\begin{bmatrix}I_{n_E} & -\bar{A}_{12}\bar{A}_{22}^{-1} & 0\\ 0 & I_{n-n_E} & 0\\ 0 & 0 & I_q\end{bmatrix},\\ M_2&=\begin{bmatrix}I_{n_E} & 0 & 0\\ -\bar{A}_{22}^{-1}\bar{A}_{21} & I_{n-n_E} & -\bar{A}_{22}^{-1}\bar{E}_2\\ 0 & 0 & I_q\end{bmatrix}\end{aligned}$$

These two matrices are clearly nonsingular. A straightforward calculation shows that

$$M_1\begin{bmatrix}\bar{\mathbf{E}} & 0\\ 0 & I_q\end{bmatrix}M_2=M_1\begin{bmatrix}I_{n_E} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I_q\end{bmatrix}M_2=\begin{bmatrix}I_{n_E} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I_q\end{bmatrix}$$

$$M_1\begin{bmatrix}\bar{A} & \bar{E}\\ 0 & A_1\end{bmatrix}M_2=M_1\begin{bmatrix}\bar{A}_{11} & \bar{A}_{12} & \bar{E}_1\\ \bar{A}_{21} & \bar{A}_{22} & \bar{E}_2\\ 0 & 0 & A_1\end{bmatrix}M_2=\begin{bmatrix}A_r & 0 & E_r\\ 0 & \bar{A}_{22} & 0\\ 0 & 0 & A_1\end{bmatrix},$$

$$\begin{bmatrix}\bar{C} & F\end{bmatrix}M_2=\begin{bmatrix}\bar{C}_1 & \bar{C}_2 & F\end{bmatrix}M_2=\begin{bmatrix}C_r & \bar{C}_2 & F_r\end{bmatrix}$$

One can easily verify the strong stabilizability of

$$\left(\begin{bmatrix} I_{n_E} & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_r & E_r & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & \bar{A}_{22} \end{bmatrix}^T, \begin{bmatrix} C_r & F_r & \bar{C}_2 \end{bmatrix}^T\right)$$

which, in turn, implies the delectability of $\left(\begin{bmatrix} A_r & E_r \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} C_r & F_r \end{bmatrix}\right)$.

By Remark 3.3, the output regulation problem for system (3.14) is solvable by a normal output controller, which will be denoted as

$$\begin{cases} u = k(z(t)) \\ \dot{z}(t) = g(z(t), y(t)) \end{cases} \tag{3.19}$$

where $z \in \Re^{n_z}$ .

*Step 3:* We now show that the controller (3.19) will also solve the output regulation problem for the original system (3.1), or that the closed-loop system composed of (3.1) and (3.19) satisfies P1 and P2. To this end, let the linearization of (3.19) be

$$\begin{cases} u = K^* z \\ \dot{z}(t) = G_1 z + G_2 y \end{cases} \tag{3.20}$$

Let $A_c$ be the Jacobian of the closed-loop system composed of (3.14) and (3.19), and $(\bar{\mathbf{E}}_c, \bar{A}_c)$ be the linearization of the closed-loop system composed of (3.1) and (3.19). Then we have

$$A_c = \begin{bmatrix} A_r & B_r K^* \\ G_2 C_r & G_1 + G_2 R K^* \end{bmatrix}$$

and

$$\begin{aligned} \bar{\mathbf{E}}_c &= diag\{\mathbf{E}, I_{n_z}\} \\ \bar{A}_c &= \begin{bmatrix} A & BK^* \\ G_2 C & G_1 \end{bmatrix} \end{aligned}$$

Let

$$\begin{aligned} N_1 &= \begin{bmatrix} I_{n_E} & -\bar{A}_{12}\bar{A}_{22}^{-1} & 0 \\ 0 & 0 & I_{n_z} \\ 0 & I_{n-n_E} & 0 \end{bmatrix}, \\ N_2 &= \begin{bmatrix} I_{n_E} & 0 & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & -\bar{A}_{22}^{-1}\bar{B}_2 K^* & I_{n-n_E} \\ 0 & I_{n_z} & 0 \end{bmatrix} \end{aligned}$$

A simple calculation shows that

$$
\begin{aligned}
& N_1 \begin{bmatrix} T_1 & 0 \\ 0 & I_{n_z} \end{bmatrix} \bar{A}_c \begin{bmatrix} T_2 & 0 \\ 0 & I_{n_z} \end{bmatrix} N_2 = N_1 \begin{bmatrix} \bar{A} & \bar{B}K^* \\ G_2\bar{C} & G_1 \end{bmatrix} N_2 \\
= & N_1 \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 K^* \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_2 K^* \\ G_2\bar{C}_1 & G_2\bar{C}_2 & G_1 \end{bmatrix} N_2 = \begin{bmatrix} A_r & B_r K^* & 0 \\ G_2 C_r & G_1 + G_2 R K^* & C_2\bar{C}_2 \\ 0 & 0 & \bar{A}_{22} \end{bmatrix} \\
= & \begin{bmatrix} A_c & \begin{bmatrix} 0 \\ C_2\bar{C}_2 \end{bmatrix} \\ 0 & \bar{A}_{22} \end{bmatrix} \\
& N_1 \begin{bmatrix} T_1 & 0 \\ 0 & I_{n_z} \end{bmatrix} \bar{\mathbf{E}}_c \begin{bmatrix} T_2 & 0 \\ 0 & I_{n_z} \end{bmatrix} N_2 = N_1 \begin{bmatrix} \bar{\mathbf{E}} & 0 \\ 0 & I_{n_z} \end{bmatrix} N_2 \\
= & N_1 \begin{bmatrix} I_{n_E} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_z} \end{bmatrix} N_2 = \begin{bmatrix} I_{n_E} & 0 & 0 \\ 0 & I_{n_z} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{n_E+n_z} & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

from which one can easily verify that the stability of $A_c$ and nonsingularity of $\bar{A}_{22}$ together imply the strong stability of $(\bar{\mathbf{E}}_c, \bar{A}_c)$.

Finally, to verify that P2 is satisfied, one only needs to note that

$$
\begin{aligned}
0 &= \lim_{t\to\infty} h_r\left(\bar{x}_1(t), u(t), v(t)\right) = \lim_{t\to\infty} \bar{h}\left(\bar{x}(t), v(t)\right) \\
&= \lim_{t\to\infty} h\left(x(t), v(t)\right)
\end{aligned}
$$

□

**Remark 3.6:** The solution of the closed-loop system, composed of the original plant (3.1) and the controller (3.19), can be given in terms of the closed-loop system, composed of the reduced-order system and (3.19), as follows:

$$
x(t) = T_2\bar{x}(t) = \begin{cases} T_2 \begin{bmatrix} \bar{x}_1 \\ \alpha(\bar{x}_1, k(z), v) \end{bmatrix}, t > 0 \\ x_0, t = 0 \end{cases}
$$

which clearly shows the strongly stability of the closed-loop system, and exhibits the impulse-free nature of the closed-loop system [24].

**Remark 3.7:** Condition (3.9) is the key to the validity of Lemma 3.5. This condition of course makes the lemma less appealing. What makes our approach interesting is that this condition can actually be removed through an output feedback precompensator, as shown in the following lemma.

**Lemma 3.8:** Under H2 and H3, there exists a linear output feedback control

$$u = Ky + u' \tag{3.21}$$

such that the system with $u'$ as an input, *i.e.*,

$$\begin{cases} \mathbf{E}\dot{x} = \tilde{f}\,(x, u', v) = f(x, Kh(x,v) + u', v) \\ y = \tilde{h}(x,v) = h(x,v) \end{cases} \tag{3.22}$$

satisfies:

i) $\deg\left(\det\left(\lambda\mathbf{E} - \tilde{A}\right)\right) = n_E$, where $\tilde{A}$ is the Jacobian of $\tilde{f}\,(x, u', v)$ ;

ii) the linearization of (3.22) is strongly stabilizable, and strongly detectable.

Moreover, assume that there exist sufficiently smooth functions $\mathbf{x}(v)$ and $\mathbf{u}(v)$, with $\mathbf{x}(0) = 0$ and $\mathbf{u}(0) = 0$, both defined in a neighborhood $V$ of the origin of $\Re^q$, satisfying equation (3.6). Then the pair $(\mathbf{x}(v), \mathbf{u}(v))$ is also the solution of the regulator equations associated with (3.22) and (3.2).

*Proof:* Under H2, there exists a matrix $L$ satisfying

$$\deg\left(\det\left(\lambda\mathbf{E} - (A + BL)\right)\right) = n_E$$

Using the same coordinate transformation $T_1, T_2$, as used in Lemma 3.5, gives

$$\deg\left(\det\left(\begin{bmatrix} \lambda I_{n_E} - \bar{A}_{11} - \bar{B}_1\bar{L}_1 & -\bar{A}_{12} - \bar{B}_1\bar{L}_2 \\ -\bar{A}_{21} - \bar{B}_2\bar{L}_1 & -\bar{A}_{22} - \bar{B}_2\bar{L}_2 \end{bmatrix}\right)\right) = n_E$$

where $[\bar{L}_1, \bar{L}_2] = LT_2$. Hence, $\det(\bar{A}_{22} + \bar{B}_2\bar{L}_2) \neq 0$, *i.e.*, the pair $(\bar{A}_{22}, \bar{B}_2)$ is normalizable.

Similarly, under H3, there exist matrices $G_1$ and $G_2$ satisfying

$$\deg\left(\det\left(\lambda\begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_q \end{bmatrix} - \left(\begin{bmatrix} A & E \\ 0 & A_1 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\begin{bmatrix} C & F \end{bmatrix}\right)\right)\right) = n_E + q$$

which gives

$$\deg\left(\det\left(\begin{bmatrix} \lambda I_{n_E} - \bar{A}_{11} + \bar{G}_1\bar{C}_1 & -\bar{A}_{12} + \bar{G}_1\bar{C}_2 & -\bar{E}_1 + \bar{G}_1F \\ -\bar{A}_{21} + \bar{G}_2\bar{C}_1 & -\bar{A}_{22} + \bar{G}_2\bar{C}_2 & -\bar{E}_2 + \bar{G}_2F \\ G_2\bar{C}_1 & G_2\bar{C}_2 & \lambda I_q - A_1 + G_2F \end{bmatrix}\right)\right) = n_E + q$$

where $\begin{bmatrix} \bar{G}_1 \\ \bar{G}_2 \end{bmatrix} = T_1 G_1$. Hence, $\det(\bar{A}_{22} - \bar{G}_2 \bar{C}_2) \neq 0$, *i.e.*, the pair $\left(\bar{A}_{22}^T, \bar{C}_2^T\right)$ is normalizable. (In fact, it suffices to assume the strong detectability of $(\mathbf{E}, A, C)$ to obtain this normalizability property.)

By Lemma 3-5.1 of [10], the normalizability of $(\bar{A}_{22}, \bar{B}_2)$ and the normalizability of $\left(\bar{A}_{22}^T, \bar{C}_2^T\right)$ together guarantee the existence of a matrix $K \in \Re^{m \times p}$ such that

$$\det\left(\bar{A}_{22} + \bar{B}_2 K \bar{C}_2\right) \neq 0$$

Let the linearization of $\tilde{f}\,(x, v, v)$ be

$$\mathbf{E}\dot{x} = (A + BKC)x + Bv + (E + BKF)v = \tilde{A}x + \tilde{B}v + \tilde{E}v$$

Then the fulfillment of (i) follows from

$$\deg\left(\det\left(\lambda\mathbf{E} - \tilde{A}\right)\right) = \deg\left(\det\left(\begin{bmatrix} \lambda I_{n_E} - \bar{A}_{11} - \bar{B}_1 K \bar{C}_1 & -\bar{A}_{12} - \bar{B}_1 K \bar{C}_2 \\ -\bar{A}_{21} - \bar{B}_2 K \bar{C}_1 & -\bar{A}_{22} - \bar{B}_2 K \bar{C}_2 \end{bmatrix}\right)\right) = n_E$$

To verify the fulfillment of (ii), one only needs to note the simple fact that, for any matrices $k \in \Re^{m \times n}$ and $g \in \Re^{(n+q) \times p}$, $(\mathbf{E}, A + Bk)$ is strongly stable if and only if $\left(\mathbf{E}, \tilde{A} + \tilde{B}(k - KC)\right)$ is so; and $\left(\begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & A_1 \end{bmatrix} - g \begin{bmatrix} C & F \end{bmatrix}\right)$ is strongly stable if and only if

$$\left(\begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} \tilde{A} & \tilde{E} \\ 0 & A_1 \end{bmatrix} - \begin{bmatrix} g_1 + BK \\ g_2 \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix}\right)$$

is so, where $g^T = (g_1^T, g_2^T)$.

Finally, suppose $\mathbf{x}(v)$ and $\mathbf{u}(v)$ are the solution of the regulator equations associated with (3.6). Then

$$\tilde{f}\,(\mathbf{x}(v), \mathbf{u}(v), v) = f(\mathbf{x}(v), Ke + \mathbf{u}(v), v) =$$

$$f(\mathbf{x}(v), Kh(\mathbf{x}(v), v) + \mathbf{u}(v), v) = f(\mathbf{x}(v), \mathbf{u}(v), v)$$

Thus, $\mathbf{x}(v)$ and $\mathbf{u}(v)$ are also the solution of the regulator equations associated with (3.22) and (3.2). □

Combining Lemmas 3.5 and 3.8 leads to our main result as follows:

**Theorem 3.9:** Under hypotheses H1 to H3, the output regulation problem of system (3.1) and (3.2) *via* a *normal output feedback controller* is solvable if and only if there exist sufficiently smooth functions $\mathbf{x}(v)$ and $\mathbf{u}(v)$, with $\mathbf{x}(0) = 0$ and $\mathbf{u}(0) = 0$, both defined in a neighborhood $V$ of the origin of $\Re^q$, satisfying equation (3.6).

**Remark 3.10:** The matrix $K$ in Lemma 3.8 can actually be constructed as follows [10]. Denote $rank\,(\bar{A}_{22}) = s$. If $s = n - n_E$, then $\bar{A}_{22}$ is nonsingular and Lemma 3.5 suffices to solve the problem. Otherwise, suppose $s < n - n_E$. Then there exist two nonsingular matrices $Q, P \in \Re^{(n-n_E)\times(n-n_E)}$, such that

$$Q\bar{A}_{22}P = diag\,(I_s, 0)\,, QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, CP = [C_1, C_2]$$

where $B_1 \in \Re^{s\times m}, C_1 \in \Re^{p\times s}$. In this case, we can choose $K = B_2^T C_2^T$.

**Remark 3.11:** The way we construct the controller is completely different from the one given in [22], where the authors employed the normalizability decomposition that leads to a reduced-order *singular* nonlinear system. Success of that approach relies on the solvability of the output regulation problem for this reduced-order *singular* nonlinear system. So far, this technique only works for the systems that would lead to a reduced-order singular system whose error output equation does not depend on $u$. This is the case when $f(x, u, v)$ is linear in $u$ with a constant input gain, as described in Remark 3.4. In contrast, the technique in this chapter employs a standard coordinate transformation, well-known in the literature of singular systems [10]. Success of this approach depends on that this transformation can lead to a well-defined *normal* nonlinear system for which the output regulation problem is solvable. As established in Lemmas 3.5 and 3.8, this objective is always achievable provided that a linear precompensator described in Lemma 3.8 is first employed. Therefore, our result applies to general singular nonlinear systems. We note that our approach also relies on the fact that the output regulation theory for normal nonlinear systems can handle the systems whose error output equation depends on $u$, as described in Remark 3.3.

## 3.4 An Example

Consider the following singular nonlinear system:

$$\begin{cases} \dot{x}_1 = -2x_3 - 3v_1 + v_2 + u + x_3^2 + v_1 u - v_1 v_2 \\ 0 = x_2 - v_1 + v_2 + x_1^2 \\ 0 = -x_1 + u + 2v_1 \\ 0 = x_4 + v_2 + x_3^2 + x_1^2 - v_1^2 \\ y = 2x_1 + x_3 - v_1 - 2v_2 \end{cases} \tag{3.23}$$

with the exosystem

$$\dot{v}_1 = 2v_2, \dot{v}_2 = -2v_1$$

which is not linear in $u$ with a constant input gain.

Linearizing (3.23) gives

$$\begin{aligned} \mathbf{E} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} 2 & 0 & 1 & 0 \end{bmatrix}, E = \begin{bmatrix} -3 & 1 \\ -1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}, F = \begin{bmatrix} -1 & -2 \end{bmatrix} \end{aligned}$$

Simple calculation shows that neither is $(\mathbf{E}, B)$ normalizable nor is condition (3.9) satisfied. Nevertheless, Hypotheses H1 to H3 are satisfied, and the regulator equations have the following unique solution:

$$\begin{aligned} \mathbf{x}_1(v) &= v_1 + v_2 \\ \mathbf{x}_2(v) &= v_1 - v_2 - (v_1 + v_2)^2 \\ \mathbf{x}_3(v) &= -v_1 \\ \mathbf{x}_4(v) &= -v_2 - (v_1 + v_2)^2 \\ \mathbf{u}(v) &= -v_1 + v_2 \end{aligned}$$

Thus, our approach applies to this system. In fact, using a precompensator $u = y + u'$

gives the following:

$$\begin{cases} \dot{x}_1 = -x_3 + 2x_1 + u' - 4v_1 - v_2 + x_3^2 + v_1\left(2x_1 + x_3 - v_1 - 3v_2 + u'\right) \\ 0 = x_2 - v_1 + v_2 + x_1^2 \\ 0 = x_1 + x_3 + u' + v_1 - 2v_2 \\ 0 = x_4 + v_2 + x_3^2 + x_1^2 - v_1^2 \end{cases} \tag{3.24}$$

which satisfies all conditions of Lemma 3.5. Thus, we can solve $x_2, x_3$ and $x_4$ in terms of $x_1$, $v_1$, $v_2$, and $v$ from (3.24), as follows:

$$\begin{cases} x_2 = v_1 - v_2 - x_1^2 \\ x_3 = -x_1 - u' - v_1 + 2v_2 \\ x_4 = -v_2 - \left(x_1 + u' + v_1 - 2v_2\right)^2 - x_1^2 + v_1^2 \end{cases} \tag{3.25}$$

Substituting (3.25) into (3.24) gives a reduced-order normal system as follows:

$$\begin{cases} \dot{x}_1 = 3x_1 + 2u' - 3v_1 - 3v_2 + \left(x_1 + u' + v_1 - 2v_2\right)^2 + v_1\left(3x_1 - 2v_1 - v_2\right) \\ y = x_1 - u' - 2v_1 \end{cases} \tag{3.26}$$

We are now ready to design a controller to solve the output regulation problem of the normal system (3.26), according to the method in [27] and [23], as follows:

$$\begin{cases} u' = z_3 - z_2 - 2\left(z_1 - z_2 - z_3\right) \\ \dot{z}_1 = 3z_1 + 2u' - 3z_2 - 3z_3 + \left(z_1 + u' + z_2 - 2z_3\right)^2 \\ +z_2\left(3z_1 - 2z_2 - z_3\right) - 7\left(z_1 - u' - 2z_2 - y\right) \\ \dot{z}_2 = 2z_3 + \left(z_1 - u' - 2z_2 - y\right) \\ \dot{z}_3 = -2z_2 + \left(z_1 - u' - 2z_2 - y\right) \end{cases} \tag{3.27}$$

Composition of this controller with the precompensator $u = y + u'$ gives a normal output feedback controller that solves the output regulation problem of the original system. A simple simulation result is shown in Figure 3.1.

## 3.5 Concluding Remarks

This chapter has removed the normalizability assumption for general singular nonlinear systems, thus leading to a complete solution to the output regulation problem of singular nonlinear systems *via* normal output feedback. The result has bridged the gap between the linear case and the nonlinear case, and is practically useful due to the advantage of normal controllers over singular controllers. An example that cannot be handled by existing approaches has been used to show the solvability and effectiveness of this approach.

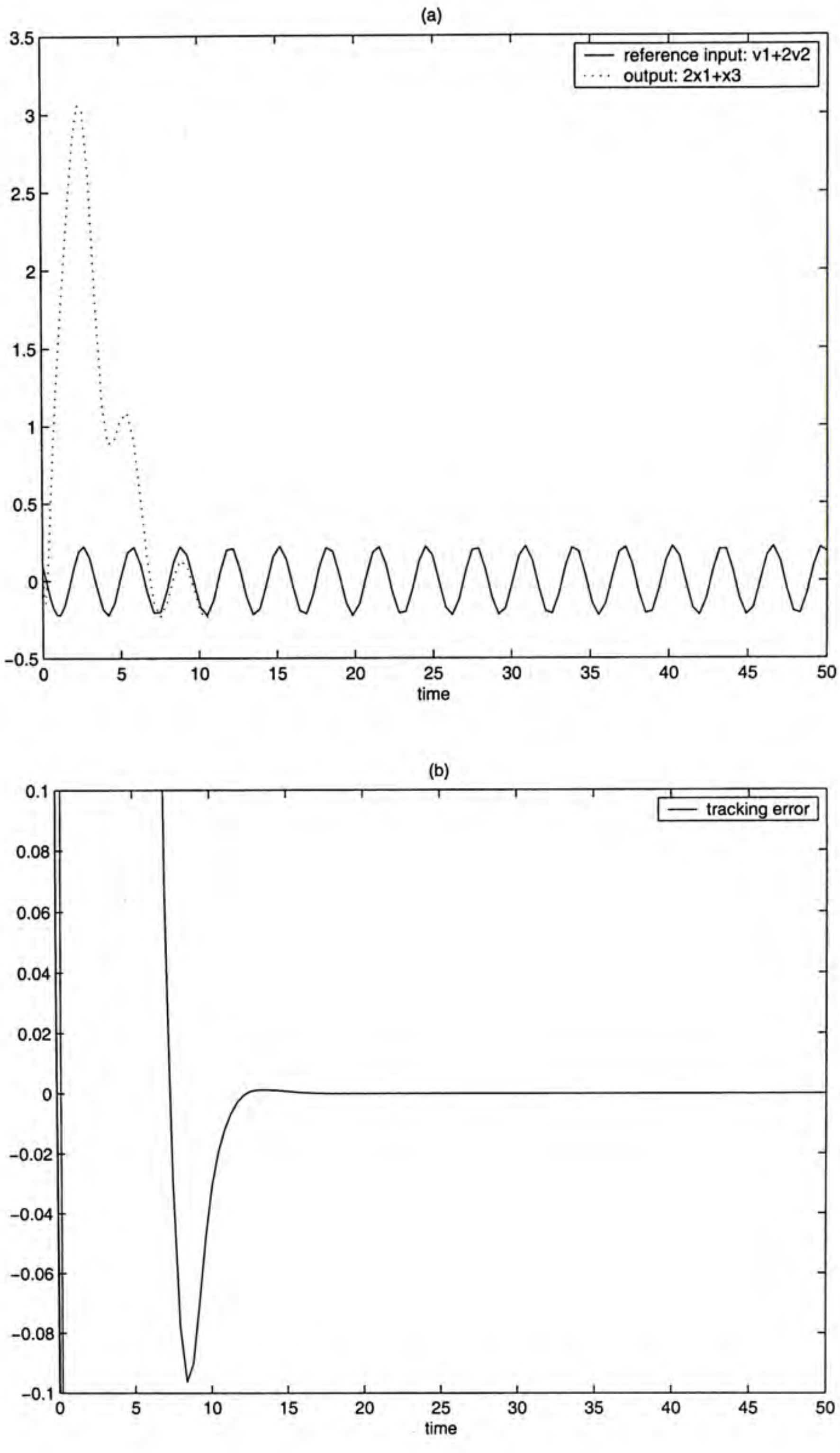


Figure 3.1: (a): the system trajectories, and (b): the profile of the tracking error

# Chapter 4

# Robust Output Regulation of Singular Nonlinear Systems

The output regulation problem for singular nonlinear systems has been studied recently for the ideal case where the mathematical model is exactly known. We will further consider the robust output regulation problem for the same class of singular nonlinear systems which may contain uncertain parameters. And we will establish some conditions for the solvability of the problem, thus extending the existing results from the normal nonlinear systems to the singular nonlinear systems.

This chapter is organized as follows. Section 4.1 gives an introduction. Section 4.2 describes the problem and summarizes some results obtained in [24]. And Section 4.3 gives a preliminary result, which solves the $k^{th}$-order robust output regulation problem. The main result of this chapter is given in Section 4.4, where we will derive the solvability conditions for the problem, and construct the controller. The result can be viewed as a generalization of the same problem for the normal nonlinear systems studied in [17].

## 4.1 Introduction

In the last chapter, we have investigated an open problem on output regulation of singular nonlinear systems without considering model uncertainty. Here, we focus on the robust version, *i.e.*, the controller is further required being able to tolerate certain plant uncertainty.

For linear systems, this robust version of output regulation problem was thoroughly studied for normal systems in the 1970s, in [13], [14], among others. A salient outcome of these research activities is the internal model principle, which is an extension of the well-known PID control. The problem was also investigated for linear singular systems in the 1980s [10]. Recently, a more clear-cut solution of this problem for linear singular systems was obtained in [34]. For nonlinear systems, the output regulation problem was first treated for normal systems. The special case in which the exogenous signals are constant was studied in [14], [22]. The general case with time-varying exogenous signals was studied in [27] without considering parameter uncertainty. Subsequently, the robust version of the same problem was pursued in [3], [17], [19], [20]. The objective of this chapter is to further pursue the research initiated in [24] by considering the presence of plant uncertainty, so as to obtain a solution of the robust output regulation problem for singular nonlinear systems.

## 4.2 Problem Description and Standard Assumptions

Consider the singular system (3.1) with uncertain parameters, described by

$$\begin{cases} \mathbf{E}\dot{x}(t) = f(x(t), u(t), v(t), w), x(0) = x_0 \\ y(t) = h(x(t), v(t), w), t \geq 0 \end{cases} \tag{4.1}$$

with same exosystem as (3.2):

$$\dot{v}(t) = a(v(t)), v(0) = v_0 \tag{4.2}$$

where $x(t) \in \Re^n$ is the plant state, $u(t) \in \Re^m$ the plant input, $y(t) \in \Re^p$ the plant output representing the tracking error, $v(t) \in \Re^q$ the exogenous signal representing the disturbance and/or the reference input, $w \in \Re^N$ the plant unknown parameters, and $\mathbf{E} \in \Re^{n \times n}$ a singular constant matrix, with $rank\,(\mathbf{E}) = n_E < n$. Also, it is assumed that 0 is the nominal value of the uncertain parameters $w$.

This chapter will focus on the dynamic output feedback controller described by

$$\begin{cases} u\,(t) = k(z(t), y(t)) \\ \dot{z}(t) = g(z(t), y(t)) \end{cases} \tag{4.3}$$

where $z(t)$ is the compensator state vector of dimension $n_c$. When the system state is available, a state feedback control law can be considered accordingly.

The closed-loop system, composed of plant (4.1), exosystem (4.2) and control law (4.3), can be put into the following form:

$$\begin{cases} \mathbf{E}_c\dot{x}_c(t) = f_c(x_c(t), v(t), w), x_c(0) = x_{c0} \\ \dot{v} = a(v(t)), v(0) = v_0 \\ y(t) = h_c(x_c(t), v(t), w) \end{cases} \tag{4.4}$$

where

$$\begin{cases} x_c = \begin{bmatrix} x \\ z \end{bmatrix}, \mathbf{E}_c = \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_{n_c} \end{bmatrix}, h_c(x_c, v, w) = h(x, v, w) \\ f_c(x_c, v, w) = \begin{bmatrix} f(x, k(z, h(x, v, w)), v, w) \\ g(z, h(x, v, w)) \end{bmatrix} \end{cases} \tag{4.5}$$

Throughout this chapter, it is assumed that all the functions involved in this setup are sufficiently smooth and defined globally on appropriate Euclidean spaces, and $a(0) = 0$, $f(0,0,0,w) = 0$, and $h(0,0,0,w) = 0$ for any $w \in W$ with $W$ being an open neighborhood of the origin of $\Re^N$. Our results will be stated locally in terms of $V$ and $W$ with $V$ being an open neighborhood of the origin in $\Re^q$. In the sequel, $V$ and $W$ are implicitly permitted to be made small so as to accommodate subsequent local arguments.

Using the following notation,

$$A(w) = \frac{\partial f}{\partial x}|_{x=u=v=0}, B(w) = \frac{\partial f}{\partial u}|_{x=u=v=0},$$
$$E(w) = \frac{\partial f}{\partial v}|_{x=u=v=0}, C(w) = \frac{\partial h}{\partial x}|_{x=u=v=0},$$
$$F(w) = \frac{\partial h}{\partial v}|_{x=u=v=0}, A_1 = \frac{\partial a(v)}{\partial v}|_{v=0}, A_c(w) = \frac{\partial f_c}{\partial x_c}|_{x_c=v=0}$$

system (4.1) and (4.2) can be rewritten as

$$\begin{cases} \mathbf{E}\dot{x} = A(w)x + B(w)u + E(w)v + o(x, u, v, w) \\ y = C(w)x + F(w)v + o(x, u, v, w) \\ \dot{v} = A_1 v + o(v) \end{cases}$$

where $o(x, u, v, w)$ (or $o(v)$) is a sufficiently smooth function vanishing at $(x, u, v) = (0, 0, 0)$ (or $v = 0$) together with its first-order derivative, for any $w \in W$. For convenience, let $A, B, \cdots$, denote $A(0), B(0), \cdots$, respectively.

*The robust output regulation problem:* Find a control law such that the closed-loop (4.4) has the following two properties:

(P1) The linearization at $x_c = 0$ of

$$\mathbf{E}_c\dot{x}_c(t) = f_c(x_c(t), 0, 0)$$

is strongly stable in the sense described in (P1) of Chapter 3.

(P2) The trajectories starting from all sufficiently small initial states $(x_{c0}, v_0)$ satisfy

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} h_c(x_c(t), v(t), w) = 0 \tag{4.6}$$

**Remark 4.1:** The above problem is clearly an extension of the problem studied in [24] by taking into account the uncertainty. Viewing $w$ as generated by an exosystem of the form $\dot{w} = 0$, a solvability condition can be obtained, by slightly modifying Lemma 4.1 of [24], as follows:

**Theorem 4.2:** Assume the following:

(H1): $v = 0$ is a stable equilibrium of the exosystem (4.2), and there exists a neighborhood $V$ of the origin of $\Re^q$ with the property that each initial condition $v_0 \in V$ is stable in the sense of Poisson.

Then, the controller (4.3) solves the robust output regulation problem for the singular uncertain system (4.1) and (4.2) if it is such that the closed-loop system satisfies the following:

(i) (P1) holds;

(ii) there exists a sufficiently smooth function $\mathbf{x}_c(v, w)$, locally defined in $V \times W$, satisfying $\mathbf{x}_c(0,0) = 0$ and

$$\begin{cases} \mathbf{E}_c \frac{\partial \mathbf{x}_c(v,w)}{\partial v} a(v) = f_c(\mathbf{x}_c(v,w), v, w) \\ h_c(\mathbf{x}_c(v,w), v, w) = 0 \end{cases} \tag{4.7}$$

To close this section, some standard assumptions are listed:

(H2): $(\mathbf{E}, A, B)$ is strongly stabilizable, *i.e.*, there exists a matrix $K \in \Re^{m\times n}$ such that $(\mathbf{E}, A + BK)$ is strongly stable.

(H3): $(\mathbf{E}, A, C)$ is strongly detectable, *i.e.*, there exists a matrix $K \in \Re^{n\times p}$ such that $(\mathbf{E}, A + KC)$ is strongly stable.

(H4): There exist sufficiently smooth functions $\mathbf{x}(v, w)$ and $\mathbf{u}(v, w)$, with $\mathbf{x}(0,0) = 0$ and $\mathbf{u}(0,0) = 0$, such that, for $v \in V, w \in W$

$$\begin{cases} \mathbf{E} \frac{\partial \mathbf{x}(v,w)}{\partial v} a(v) = f(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w) \\ h(\mathbf{x}(v,w), v, w) = 0 \end{cases} \tag{4.8}$$

**Remark 4.3:** When $\mathbf{E}$ is an identity matrix, Assumptions H2 and H3 are reduced to exactly the same ones assumed by [27] for *normal* systems. Equations (4.8) become the so-called regulator equations discovered by Isidori and Byrnes [27].

## 4.3 A Preliminary Result

Recall that, in the normal case, the way to handle the robust output regulation problem is much more complicated than the way to handle the output regulation problem. This is because, when there is no uncertainty, the solution of equation (4.8) or its estimation can be used as a feedforward function to cancel the steady state error output. But this is impossible when the uncertain parameter vector $w$ is presented since the solution of equation (4.8) also depends on $w$, which however cannot appear in the control law. As a result, the problem has to be approached with the employment of a nonlinear version of the internal model principle [17], [19], [20]. Here, this technique will be further extended to singular nonlinear systems.

To begin with, some notations that have been used frequently in [17], [19], [20] are first introduced. For any matrix $M$, define

$$M^{(0)} = I, M^{(1)} = M, \cdots, M^{(k)} = \underbrace{M \otimes \cdots \otimes M}_{k\ factors}, k = 1, 2, \cdots$$

where $\otimes$ denotes the Kronecker product. Also let $v^{[l]}$ denote the vector

$$v^{[l]} = [v_1^l, v_1^{l-1}v_2, \cdots, v_1^{l-1}v_q, v_1^{l-2}v_2^2, v_1^{l-2}v_2v_3, \cdots, v_1^{l-2}v_2v_q, \cdots, v_q^l]^T \tag{4.9}$$

**Remark 4.4:** It was shown in [17] that if $v$ satisfies $\dot{v} = A_1 v$ for some square matrix $A_1$, then there exist square matrices $A_l$, $l = 2, 3, \cdots$, such that

$$\dot{v}^{[l]}(t) = A_l v^{[l]}(t), \ l = 2, 3, \cdots$$

In fact, $A_l$ can be explicitly given as

$$A_l = M_l \left[ \sum_{i=1}^{l} I_q^{(i-1)} \otimes A_1 \otimes I_q^{(l-i)} \right] N_l$$

where $M_l$ and $N_l$ are such that $v^{[l]} = M_l v^{(l)}, v^{(l)} = N_l v^{[l]}$, and $I_q$ denote the $q$-dimensional identity matrix. As in [17], the following autonomous system is called the K-fold exosystem:

$$\begin{bmatrix} \dot{v}^{[1]} \\ \dot{v}^{[2]} \\ \vdots \\ \dot{v}^{[k]} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} \begin{bmatrix} v^{[1]} \\ v^{[2]} \\ \vdots \\ v^{[k]} \end{bmatrix} \tag{4.10}$$

A linear result, which will play an important role in establishing our main result, is first stated.

**Lemma 4.5:** Given any square matrix $\hat{A}_1$, with all the eigenvalues located on the closed right complex plane, let $\beta_i \in \Re^{n_i \times n_i}, \sigma_i \in \Re^{1 \times n_i}$, $i = 1, \cdots, r$, for some positive integers $n_1, \cdots, n_r, r$, such that

(i) the pair $(\beta_i, \sigma_i)$ is controllable, and

(ii) the minimal polynomial of $\hat{A}_1$ divides the characteristic polynomial of $\beta_i$.

Also, let

$$G_1 = block\ diag\ [\beta_1, \cdots, \beta_r],$$

$$G_2 = block\ diag\ [\sigma_1, \cdots, \sigma_r]$$

and finally, let matrices $g_1$ and $g_2$ have the following form:

$$g_1 = T \begin{bmatrix} S_1 & S_2 \\ 0 & G_1 \end{bmatrix} T^{-1}, \quad g_2 = T \begin{bmatrix} S_3 \\ G_2 \end{bmatrix}$$

where $S_1, S_2, S_2$ are arbitrary matrices with proper dimensions, and $T$ is a nonsingular matrix. Then, for any matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ with appropriate dimensions, if the matrix

$$\begin{bmatrix} \hat{A} & \hat{B} \\ g_2\hat{C} & g_1 + g_2\hat{D} \end{bmatrix} \tag{4.11}$$

is Hurwitz, then for any $(U, V)$ with proper dimensions, the linear matrix equation

$$\begin{cases} \phi\hat{A}_1 = \hat{A}\phi + \hat{B}\theta + U \\ \theta\hat{A}_1 = g_1\theta + g_2(\hat{C}\phi + \hat{D}\theta + V) \end{cases} \tag{4.12}$$

has a unique solution which satisfies

$$\hat{C}\phi + \hat{D}\theta + V = 0 \tag{4.13}$$

*Proof:* Since (4.12) is a Sylvester equation, it follows from the assumptions on matrices $\hat{A}_1$ and (4.11) that equation (4.12) has a unique solution. To verify (4.13), let $\theta = [\hat{\theta}^T, \bar{\theta}^T]^T$, where $\bar{\theta}$ has the same dimension as $G_1$. Then the second equation of (4.12) implies

$$\bar{\theta}\hat{A}_1 - G_1\bar{\theta} = G_2Y \tag{4.14}$$

where $Y = \hat{C}\phi + \hat{D}\theta + V$.

Due to the block diagonal structure of $G_1, G_2$, we can assume $r = 1$ without loss of generality. Consequently, we can write $G_1, G_2$ in the following form:

$$G_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_{n_k} & -\alpha_{n_k-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix}, G_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let $\theta_j, j = 1, \cdots, n_k$, and denote the $j^{th}$ row of $\bar{\theta}$. Then expanding (4.14) gives

$$\begin{bmatrix} \theta_1 \hat{A}_1 - \theta_2 \\ \theta_2 \hat{A}_1 - \theta_3 \\ \vdots \\ \theta_{n_k-1} \hat{A}_1 - \theta_{n_k} \\ \theta_{n_k} \hat{A}_1 + \alpha_{n_k} \theta_1 + \cdots + \alpha_1 \theta_{n_k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Y \end{bmatrix}$$

Furthermore,

$$Y = \theta_1 (\hat{A}_1^{n_k} + \alpha_1 \hat{A}_1^{n_k-1} + \cdots + \alpha_{n_k} I)$$

The fact that the characteristic polynomial of $G_1$ is divisible by the minimal polynomial of $\hat{A}_1$ gives $Y = 0$. □

**Lemma 4.6:** Under assumption H1, for any positive integer $k$, let

$$\hat{A}_1 = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}$$

Let a linear controller of the form

$$\begin{cases} u = K_1 z + K_2 y \\ \dot{z} = g_1 z + g_2 y \end{cases} \tag{4.15}$$

be given, where $g_1$ and $g_2$ are given as in Lemma 4.5 with $r = p$. Then, if the controller (4.15) makes the pair

$$\left( \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_{n_c} \end{bmatrix}, \begin{bmatrix} A + BK_2 C & BK_1 \\ g_2 C & g_1 \end{bmatrix} \right) \tag{4.16}$$

strongly stable, then the closed-loop system composed of (4.1), (4.2) and controller (4.15) has the property that there exists a sufficiently smooth function $\mathbf{x}_c(v, w)$, locally defined in $V \times W$, satisfying $\mathbf{x}_c(0,0) = 0$ and

$$\begin{cases} \mathbf{E}_c \frac{\partial \mathbf{x}_c(v,w)}{\partial v} a(v) = f_c(\mathbf{x}_c(v,w), v, w) \\ h_c(\mathbf{x}_c(v,w), v, w) = O\left(v^{k+1}\right) \end{cases} \tag{4.17}$$

where $O\left(v^{k+1}\right)$ means that $\lim_{v \to 0} \frac{\|O(v^{k+1})\|}{\|v\|^{k+1}}$ is a constant.

*Proof:* This result can be established by performing the standard coordinate transformation on the closed-loop system. To this end, first note that the closed-loop system composed of (4.1),(4.2) and (4.15) can be written as follows:

$$\begin{cases} \mathbf{E}\dot{x} = (A(w) + B(w)K_2C(w))\, x + B(w)K_1 z \\ + (E(w) + B(w)K_2F(w))\, v + o(x, u, v, w) \\ \dot{z} = g_2 C(w)x + g_1 z + g_2 F(w)v + o(x, u, v, w) \\ y = C(w)x + F(w)v + o(x, u, v, w) \\ \dot{v} = A_1 v + o(v) \end{cases} \tag{4.18}$$

Let $T_1$ and $T_2$ be two nonsingular matrices such that $T_1 \mathbf{E} T_2 = \begin{bmatrix} I_{n_E} & 0 \\ 0 & 0 \end{bmatrix}$. Let

$$T_1 A(w) T_2 = \begin{bmatrix} A_{11}(w) & A_{12}(w) \\ A_{21}(w) & A_{22}(w) \end{bmatrix}, T_1 B(w) = \begin{bmatrix} B_1(w) \\ B_2(w) \end{bmatrix},$$

$$T_1 E(w) = \begin{bmatrix} E_1(w) \\ E_2(w) \end{bmatrix}, C(w)T_2 = \begin{bmatrix} C_1(w) & C_2(w) \end{bmatrix}, T_2^{-1} x = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

with $A_{11}(w) \in \Re^{n_E \times n_E}, B_1(w) \in \Re^{n_E \times m}, E_1(w) \in \Re^{n_E \times q}, C_1(w) \in \Re^{p \times n_E}, \bar{x}_1 \in \Re^{n_E}$, and other matrices have proper dimensions.

In terms of $\bar{x}_1$ and $\bar{x}_2$, the first equation of (4.18) can be written as

$$\begin{cases} \dot{\bar{x}}_1 = \bar{A}_{11}(w)\bar{x}_1 + \bar{A}_{12}(w)\bar{x}_2 + \bar{B}_1(w)K_1 z + \bar{E}_1(w)v + o(x, z, v, w) \\ 0 = \bar{A}_{21}(w)\bar{x}_1 + \bar{A}_{22}(w)\bar{x}_2 + \bar{B}_2(w)K_1 z + \bar{E}_2(w)v + o(x, z, v, w) \end{cases} \tag{4.19}$$

where

$$\bar{A}_{ij}(w) = A_{ij}(w) + B_i(w) K_2 C_j(w)$$

$$\bar{B}_i(w) = B_i(w)$$

$$\bar{E}_i(w) = E_i(w) + B_i(w) K_2 F(w), i, j \in \{1, 2\}$$

We will first show that $\bar{A}_{22}$ is nonsingular. In fact,

$$
\begin{aligned}
& \det\left(\begin{bmatrix} T_1 & 0 \\ 0 & I_{n_c} \end{bmatrix}\left(\lambda\begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_{n_c} \end{bmatrix} - \begin{bmatrix} A+BK_2C & BK_1 \\ g_2C & g_1 \end{bmatrix}\right)\begin{bmatrix} T_2 & 0 \\ 0 & I_{n_c} \end{bmatrix}\right) \\
= & \det\left(\begin{bmatrix} \lambda I_{n_E} - (A_{11}+B_1K_2C_1) & -(A_{12}+B_1K_2C_2) & -B_1K_1 \\ -(A_{21}+B_2K_2C_1) & -(A_{22}+B_2K_2C_2) & -B_2K_1 \\ -g_2C_1 & -g_2C_2 & \lambda I_{n_c} - g_1 \end{bmatrix}\right) \\
= & \det\left(\begin{bmatrix} \lambda I_{n_E} - \bar{A}_{11} & -\bar{A}_{12} & -\bar{B}_1K_1 \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_2K_1 \\ -g_2C_1 & -g_2C_2 & \lambda I_{n_c} - g_1 \end{bmatrix}\right) \\
= & \det(-\bar{A}_{22})\lambda^{n_E+n_c} + b(\lambda)
\end{aligned} \tag{4.20}
$$

where $b(\lambda)$ is a polynomial in $\lambda$ of degree smaller than $n_E + n_c$. It follows from the strong stability of (4.16) that $\det(\bar{A}_{22}) \neq 0$. Thus, the Implicit Function Theorem guarantees the existence of a unique sufficiently smooth solution of the second equation of (4.19), and this solution has the form

$$
\begin{aligned}
\bar{x}_2 &= \gamma(\bar{x}_1, z, v, w) = -\bar{A}_{22}^{-1}(w)\bar{A}_{21}(w)\bar{x}_1 - \bar{A}_{22}^{-1}(w)\bar{B}_2(w)K_1z \\
&- \bar{A}_{22}^{-1}(w)\bar{E}_2(w)v + o(\bar{x}_1, z, v, w)
\end{aligned} \tag{4.21}
$$

Substituting (4.21) into the first equation of (4.19), and the second and the third equations of (4.18), we obtain a reduced-order normal system as follows:

$$
\begin{cases}
\dot{\bar{x}}_1 = \bar{f}_{1c}(\bar{x}_1, z, v, w) = \hat{A}(w)\bar{x}_1 + \hat{B}(w)z + \hat{E}(w)v + o(\bar{x}_1, z, v, w) \\
\dot{z} = g_1z + g_2y \\
y = \bar{h}_c(\bar{x}_1, z, v, w) = \hat{C}(w)\bar{x}_1 + \hat{D}(w)z + \hat{F}(w)v + o(\bar{x}_1, z, v, w)
\end{cases}
$$

where

$$
\begin{aligned}
\hat{A}(w) &= \bar{A}_{11}(w) - \bar{A}_{12}(w)\bar{A}_{22}^{-1}(w)\bar{A}_{21}(w) \\
\hat{B}(w) &= \left(\bar{B}_1(w) - \bar{A}_{12}(w)\bar{A}_{22}^{-1}(w)\bar{B}_2(w)\right)K_1 \\
\hat{C}(w) &= C_1(w) - C_2(w)\bar{A}_{22}^{-1}(w)\bar{A}_{21}(w) \\
\hat{D}(w) &= \left(-C_2(w)\bar{A}_{22}^{-1}(w)\bar{B}_2(w)\right)K_1 \\
\hat{E}(w) &= \bar{E}_1(w) - \bar{A}_{12}(w)\bar{A}_{22}^{-1}(w)\bar{E}_2(w) \\
\hat{F}(w) &= F(w) - C_2(w)\bar{A}_{22}^{-1}(w)\bar{E}_2(w)
\end{aligned}
$$

Next, we will show the matrix

$$\begin{bmatrix} \hat{A} & \hat{B} \\ g_2\hat{C} & g_1+g_2\hat{D} \end{bmatrix} \tag{4.22}$$

is Hurwitz. To this end, let

$$M_1 = \begin{bmatrix} I_{n_E} & -\bar{A}_{12}\bar{A}_{22}^{-1} & 0 \\ 0 & I_{n-n_E} & 0 \\ 0 & 0 & I_{n_c} \end{bmatrix}, M_2 = \begin{bmatrix} I_{n_E} & 0 & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & I_{n-n_E} & -\bar{A}_{22}^{-1}\bar{B}_2K_1 \\ 0 & 0 & I_{n_c} \end{bmatrix}.$$

Then

$$\begin{aligned} & \det\left( M_1 \begin{bmatrix} \lambda I_{n_E} - \bar{A}_{11} & -\bar{A}_{12} & -\bar{B}_1K_1 \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_2K_1 \\ -g_2C_1 & -g_2C_2 & \lambda I_{n_c} - g_1 \end{bmatrix} M_2 \right) \\ = & \det\left( \begin{bmatrix} \lambda I_{n_E} - \hat{A} & 0 & -\hat{B} \\ 0 & -\bar{A}_{22} & 0 \\ -g_2\hat{C} & -g_2C_2 & \lambda I_{n_c} - (g_1+g_2\hat{D}) \end{bmatrix} \right) \\ = & \det\left(-\bar{A}_{22}\right) \det\left( \begin{bmatrix} \lambda I_{n_E} - \hat{A} & -\hat{B} \\ -g_2\hat{C} & \lambda I_{n_c} - (g_1+g_2\hat{D}) \end{bmatrix} \right) \end{aligned}$$

Again, it follows from the strong stability of (4.16) that the matrix (4.22) is Hurwitz. Thus, from the center manifold theorem [6], there exist sufficiently smooth functions $\bar{\mathbf{x}}_1^{(k)}(v,w)$ and $\mathbf{z}^{(k)}(v,w)$, with $\bar{\mathbf{x}}_1^{(k)}(0,0)=0$ and $\mathbf{z}^{(k)}(0,0)=0$, satisfying

$$\begin{cases} \frac{\partial \bar{\mathbf{x}}_1^{(k)}(v,w)}{\partial v}a(v) = \bar{f}_{1c}(\bar{\mathbf{x}}_1^{(k)}(v,w), \mathbf{z}^{(k)}(v,w), v, w) \\ \frac{\partial \mathbf{z}^{(k)}(v,w)}{\partial v}a(v) = g_1\mathbf{z}^{(k)}(v,w) + g_2\mathbf{y}(v,w) \\ \mathbf{y}(v,w) = \bar{h}_c(\bar{\mathbf{x}}_1^{(k)}(v,w), \mathbf{z}^{(k)}(v,w), v, w) \end{cases} \tag{4.23}$$

In terms of $v^{[l]}$ defined in (4.9), $\bar{\mathbf{x}}_1^{(k)}(v,w)$, $\mathbf{z}^{(k)}(v,w)$ and $\mathbf{y}(v,w)$ can be uniquely expressed as

$$\begin{cases} \bar{\mathbf{x}}_1^{(k)}(v,w) = \sum_{l=1}^{k}\phi_{lw}v^{[l]} + O\left(v^{k+1}\right) \\ \mathbf{z}^{(k)}(v,w) = \sum_{l=1}^{k}\theta_{lw}v^{[l]} + O\left(v^{k+1}\right) \\ \mathbf{y}(v,w) = \sum_{l=1}^{k}Y_{lw}v^{[l]} + O\left(v^{k+1}\right) \end{cases} \tag{4.24}$$

Substituting (4.24) into (4.23), and expanding (4.23) as power series in $v^{[l]}, l=1,\cdots,k$, yield the following:

$$\begin{cases} \phi_{lw}A_l = \hat{A}(w)\phi_{lw} + \hat{B}(w)\theta_{lw} + \hat{U}_{lw} \\ \theta_{lw}A_l = g_1\theta_{lw} + g_2(\hat{C}(w)\phi_{lw} + \hat{D}(w)\theta_{lw} + \hat{V}_{lw}) \\ Y_{lw} = \hat{C}(w)\phi_{lw} + \hat{D}(w)\theta_{lw} + \hat{V}_{lw} \end{cases} \tag{4.25}$$

where $\left(\hat{U}_{1w}, \hat{V}_{1w}\right) = \left(\hat{E}(w), \hat{F}(w)\right)$, and for $l = 2, 3, \cdots$, $\left(\hat{U}_{lw}, \hat{V}_{lw}\right)$ depends only on $\phi_{1w}, \cdots, \phi_{(l-1)w}$ and $\theta_{1w}, \cdots, \theta_{(l-1)w}$.

Equation (4.25) is the Sylvester equation described in Lemma 4.5, and satisfies all conditions of Lemma 4.5. Therefore, it has a unique solution for any $\left(\hat{U}_{lw}, \hat{V}_{lw}\right)$ that satisfies

$$Y_{lw} = 0, l = 1, \cdots, k \tag{4.26}$$

Thus,

$$\mathbf{y}\left(v, w\right) = \bar{h}(\bar{\mathbf{x}}_1^{(k)}\left(v, w\right), \mathbf{z}^{(k)}\left(v, w\right), v, w) = O\left(v^{k+1}\right) \tag{4.27}$$

Finally, define

$$\begin{cases} \bar{\mathbf{x}}_2^{(k)}\left(v, w\right) = \gamma(\bar{\mathbf{x}}_1^{(k)}\left(v, w\right), \mathbf{z}^{(k)}\left(v, w\right), w) \\ \mathbf{x}^{(k)}\left(v, w\right) = T_2[\bar{\mathbf{x}}_1^{(k)}\left(v, w\right)^T, \bar{\mathbf{x}}_2^{(k)}\left(v, w\right)^T]^T \\ \mathbf{x}_c(v, w) = [\mathbf{x}^{(k)}\left(v, w\right)^T, \mathbf{z}^{(k)}\left(v, w\right)^T]^T \end{cases} \tag{4.28}$$

Then, one can verify that $\mathbf{x}_c(v, w)$ satisfies (4.17) by using (4.23), (4.27), and (4.28). □

**Remark 4.7:** Consider the linear singular system of the form

$$\begin{cases} \mathbf{E}\dot{x} = Ax + Bu + Ev, \\ y = Cx + Fv \\ \dot{v} = \hat{A}_1 v \end{cases} \tag{4.29}$$

with $x \in \Re^n, u \in \Re^m, y \in \Re^p$. Assume (4.29) satisfies assumptions H2, H3, and the following:

(H5):

$$rank \begin{bmatrix} A - \mathbf{E}\lambda & B \\ C & 0 \end{bmatrix} = n + p, \forall \lambda \in \sigma\left(\hat{A}_1\right).$$

Then, there exists a controller of the form (4.15), in which $g_1$, and $g_2$ are as described in Lemma 4.5 with $r = p$ and the pair

$$\left(\begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_{n_c} \end{bmatrix}, \begin{bmatrix} A + BK_2C & BK_1 \\ g_2 C & g_1 \end{bmatrix}\right) \tag{4.30}$$

is strongly stable.

In fact, define

$$\dot{z}_1 = G_1 z_1 + G_2 y, \;\; z_1 \in \Re^{n_G} \tag{4.31}$$

where $(G_1, G_2)$ is described as in Lemma 4.5 with $r = p$, and let the dimension of $G_1$ be denoted by $n_G$. Consider the following augmented $(n + n_G)$ −dimensional system:

$$\left\{\begin{array}{l} \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_{n_G} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ G_2 C & G_1 \end{bmatrix} \begin{bmatrix} x(t) \\ z_1 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} E \\ G_2 F \end{bmatrix} v \\ \begin{bmatrix} y \\ z_1 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I_{n_G} \end{bmatrix} \begin{bmatrix} x \\ z_1 \end{bmatrix} + \begin{bmatrix} F \\ 0 \end{bmatrix} v \end{array}\right. \tag{4.32}$$

Under assumptions H2, H3, H5 and the fact that $(G_1, G_2)$ is controllable, according to the linear systems theory [10], there exists a control law of the form

$$\left\{\begin{array}{l} u = N z_2 + \begin{bmatrix} K_2 & L \end{bmatrix} \begin{bmatrix} y \\ z_1 \end{bmatrix} \\ \dot{z}_2 = A_c z_2 + \begin{bmatrix} B_{c2} & B_{c1} \end{bmatrix} \begin{bmatrix} y \\ z_1 \end{bmatrix} \end{array}\right. \tag{4.33}$$

where $z_2 \in \Re^{(n_G + n_E)}$ such that the closed-loop system is strongly stable. In fact, the controller parameters $N, L$, *etc.* can be constructed explicitly from Theorem 5-3.2 of [10]. Combining (4.31) and (4.33) gives the controller of the form (4.15), with

$$z = \begin{bmatrix} z_2 \\ z_1 \end{bmatrix}, K_1 = \begin{bmatrix} N \\ L \end{bmatrix}, g_1 = \begin{bmatrix} A_c & B_{c1} \\ 0 & G_1 \end{bmatrix}, g_2 = \begin{bmatrix} B_{c2} \\ G_2 \end{bmatrix} \tag{4.34}$$

As a result, we have the following.

**Corollary 4.8:** Under assumptions H1 to H3, assume the plant composed of (4.1) and (4.2) satisfies H5 with $\hat{A}_1 = diag\ \{A_1, \cdots, A_k\}$. Then there exists a controller of the form (4.15) such that property P1 holds and equation (4.17) is satisfied.

**Remark 4.9:** It can be shown, using the center manifold theory [6], [27], that if the closed-loop system satisfies Property P1 and equation (4.17), then for sufficiently small $(x_{c0}, v_0)$, the solution of the closed-loop system exists for all $t \geq 0$, and is bounded, and

$$\limsup_{t \to \infty} y(t) = O\left(v^{k+1}(t)\right), \quad \forall w \in W$$

For this reason, a controller that renders the closed-loop system satisfying these two properties is called a $k^{th}$-order robust regulator.

## 4.4 Solvability of the Problem

A $k^{th}$-order robust regulator is interesting in its own right since it guarantees the steady state tracking error of the closed-loop system is in the order of $k + 1$ of the exogenous signal regardless of the small variation of the uncertain parameter $w$. Moreover, under some additional assumptions, the $k^{th}$-order robust regulator actually solves the robust output regulation problem.

(H6): The exosystem is linear, that is, $a(v) = A_1 v$.

**Theorem 4.10:** Under assumptions H1 to H6, further suppose $\mathbf{u}(v, w)$ is a $k^{th}$ degree polynomial in $v$. Then the same controller that solves the $k^{th}$-order robust output regulation also solves the robust output regulation problem.

*Proof:* By the assumptions, there exists a linear control law of the form (4.15) that solves the $k^{th}$-order robust regulation problem. Clearly, the closed-loop system, composed of the plant and this control law, satisfies condition (i) of Theorem 4.2. We need to show that the closed-loop system also satisfies condition (ii) of Theorem 4.2. To this end, consider the following: system

$$\begin{cases} \mathbf{E}\dot{x} = f\left(\bar{x}, u + K_2 y, v, w\right) \\ y = h\left(x, v, w\right) \end{cases} \tag{4.35}$$

Performing the same coordinate transformation on (4.35), as what was done for the closed-loop system (4.18) in the proof of Lemma 4.6, gives

$$\begin{cases} \dot{\bar{x}}_1 = \bar{f}_1\left(\bar{x}_1, u, v, w\right) \\ y = \bar{h}\left(\bar{x}_1, u, v, w\right) \end{cases} \tag{4.36}$$

where $\bar{x}_1$ is such that $T_2^{-1}x = \left(\bar{x}_1^T, \bar{x}_2^T\right)^T$, and $\bar{f}_1\left(\bar{x}_1, u, v, w\right)$, and $\bar{h}\left(\bar{x}_1, u, v, w\right)$ are in the following form:

$$\bar{f}_1\left(\bar{x}_1, u, v, w\right) = \hat{A}(w)\bar{x}_1 + \left(\bar{B}_1(w) - \bar{A}_{12}(w)\bar{A}_{22}^{-1}(w)\bar{B}_2(w)\right)u + \hat{E}(w)v + o(\bar{x}_1, z, v, w)$$
$$\bar{h}\left(\bar{x}_1, u, v, w\right) = \hat{C}(w)\bar{x}_1 - C_2(w)\bar{A}_{22}^{-1}(w)\bar{B}_2(w)u + \hat{F}(w)v + o(\bar{x}_1, z, v, w)$$

where all the matrices in the above two equations are as defined in the proof of Lemma 4.6.

It is easy to verify that

$$\begin{cases} \bar{f}_1\left(\bar{x}_1, K_1 z, v, w\right) = \bar{f}_{1c}\left(\bar{x}_1, z, v, w\right) \\ \bar{h}\left(\bar{x}_1, K_1 z, v, w\right) = \bar{h}_c\left(\bar{x}_1, z, v, w\right) \end{cases} \tag{4.37}$$

Next, let $\mathbf{x}(v, w)$ and $\mathbf{u}(v, w)$ be the solution of (4.8). Then, clearly, $\mathbf{x}(v, w)$ and $\mathbf{u}(v, w)$ also satisfy the following:

$$\begin{cases} \mathbf{E}\frac{\partial \mathbf{x}(v,w)}{\partial v}A_1 v = f\left(\mathbf{x}(v, w), \mathbf{u}(v, w) + K_2 h(\mathbf{x}(v, w), v, w), v, w\right) \\ h(\mathbf{x}(v, w), v, w) = 0 \end{cases} \tag{4.38}$$

Let $T_2^{-1}\mathbf{x}(v,w) = (\bar{\mathbf{x}}_1^T(v,w), \bar{\mathbf{x}}_2^T(v,w))^T$. Then (4.36) and (4.38) together imply

$$\begin{cases} \frac{\partial \bar{\mathbf{x}}_1(v,w)}{\partial v} A_1 v = \bar{f}_1\left(\bar{\mathbf{x}}_1(v,w), \mathbf{u}(v,w), v, w\right) \\ \bar{h}(\bar{\mathbf{x}}_1(v,w), \mathbf{u}(v,w), v, w) = 0 \end{cases} \tag{4.39}$$

Next, we show that there exists a sufficiently smooth function $\mathbf{z}(v,w)$ such that

$$\begin{cases} \mathbf{u}(v,w) = K_1 \mathbf{z}(v,w) \\ \frac{\partial \mathbf{z}(v,w)}{\partial v} A_1 v = g_1 \mathbf{z}(v,w) \end{cases} \tag{4.40}$$

To this end, let $\bar{\mathbf{x}}_1^{(k)}(v,w)$, and $\mathbf{z}^{(k)}(v,w)$ be as defined in (4.23). Then, by (4.37) and (4.27), $\bar{\mathbf{x}}_1^{(k)}(v,w)$ and $\mathbf{z}^{(k)}(v,w)$ also satisfy the following equation:

$$\begin{cases} \frac{\partial \bar{\mathbf{x}}_1^{(k)}(v,w)}{\partial v} A_1 v = \bar{f}_1\left(\bar{\mathbf{x}}_1^{(k)}(v,w), K_1 \mathbf{z}^{(k)}(v,w), v, w\right) \\ \bar{h}(\bar{\mathbf{x}}_1^{(k)}(v,w), K_1 \mathbf{z}^{(k)}(v,w), v, w) = O(v^{k+1}) \end{cases} \tag{4.41}$$

Since $\bar{\mathbf{x}}_1^{(k)}(v,w), \mathbf{z}^{(k)}(v,w)$ take the form given by (4.24), and $\phi_{lw}$ and $\theta_{lw}$ satisfy (4.25) and (4.26), comparing (4.39) with (4.41) shows that there exist sufficiently smooth functions $\bar{\mathbf{x}}_{1k}(v,w) = O\left(v^{(k+1)}\right)$ and $\mathbf{u}_k(v,w) = O\left(v^{(k+1)}\right)$ such that

$$\begin{cases} \bar{\mathbf{x}}_1(v,w) = \sum_{l=1}^k \phi_{lw} v^{[l]} + \bar{\mathbf{x}}_{1k}(v,w) \\ \mathbf{u}(v,w) = \sum_{l=1}^k K_1 \theta_{lw} v^{[l]} + \mathbf{u}_k(v,w) \end{cases}$$

But since $\mathbf{u}(v,w)$ is assumed to be a $k^{th}$ degree polynomial in $v$, it must hold that $\mathbf{u}(v,w) = \sum_{l=1}^k K_1 \theta_{lw} v^{[l]}$. Let $\mathbf{z}(v,w) = \sum_{l=1}^k \theta_{lw} v^{[l]}$. Clearly the first equation of (4.40) is satisfied.

Now, using (4.25) and (4.26) gives $\theta_{lw} A_l = g_1 \theta_{lw}, l = 1, \cdots, k$. Hence,

$$\sum_1^k \theta_{lw} A_l v^{[l]} = \sum_1^k g_1 \theta_{lw} v^{[l]} \tag{4.42}$$

Using $\frac{\partial v^{[l]}}{\partial v} A_1 v = \dot{v}^{[l]} = A_l v^{[l]}$ in (4.42) gives

$$\frac{\partial \mathbf{z}(v,w)}{\partial v} A_1 v = \sum_{l=1}^k \theta_{lw} \frac{\partial v^{[l]}}{\partial v} A_1 v = g_1 \sum_{l=1}^k \theta_{lw} v^{[l]}$$

Thus, the second equation of (4.40) is also satisfied.

Finally, letting $\mathbf{x}_c(v,w) = (\mathbf{x}^T(v,w), \mathbf{z}^T(v,w))^T$ and substituting $\mathbf{x}_c(v,w)$ and $\mathbf{u}(v,w)$ into (4.5) give

$$h_c\left(\mathbf{x}_c(v,w), v, w\right) = h\left(\mathbf{x}(v,w), v, w\right) = 0$$

and, additionally, using (4.8) and (4.40) gives

$$\mathbf{E}_c \frac{\partial \mathbf{x}_c(v,w)}{\partial v} a(v) = \begin{bmatrix} f(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w) \\ g_1 \mathbf{z}(v,w) \end{bmatrix}$$

But (4.5) gives

$$
\begin{aligned}
f_c\left(\mathbf{x}_c(v,w),v,w\right) &= \begin{bmatrix} f(\mathbf{x}(v,w), K_1\mathbf{z}(v,w) + K_2 h\left(\mathbf{x}(v,w),v,w\right), v, w) \\ g_1\mathbf{z}(v,w) + g_2 h\left(\mathbf{x}(v,w),v,w\right) \end{bmatrix} \\
&= \begin{bmatrix} f(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w) \\ g_1\mathbf{z}(v,w) \end{bmatrix}
\end{aligned}
$$

Thus, condition (ii) of Theorem 4.2 is also satisfied. □

**Remark 4.11:** In equation (4.35), the introduction of the output feedback term $K_2 y$ is to modify the system dynamics so that the matrix $\bar{A}_{22}$, as defined in Lemma 4.6, is invertible.

## 4.5 Concluding Remarks

This chapter has extended some main results of the robust output regulation problem from linear and nonlinear normal systems to nonlinear singular systems. Even though only some sufficient conditions are given here, it is possible to establish some necessary conditions for the existence of a $k^{th}$-order robust regulator. In fact, if we assume the variations of $w$ is such that each entry of $(E(w), F(w))$ can vary arbitrarily in a neighborhood of the origin, then using a similar argument as used for normal nonlinear systems [17], it can be shown that, assumption H5 is also a necessary condition for the existence of a $k^{th}$-order robust output regulator.

# Chapter 5

# Conclusions

This thesis has addressed some important problems in nonlinear control theory including global robust stabilization of cascaded polynomial systems and output regulation of singular nonlinear systems.

Some concluding remarks are in order:

- For polynomial systems in a low-triangular form, the global robust stabilization problem can be solved under some standard hypotheses described in hypotheses H1 and H2 of Chapter 2. Furthermore, a state feedback control law in a polynomial form can be explicitly constructed if it is further assumed that the *class* $K_\infty$ function $\kappa(\cdot)$ in hypothesis H1 is a polynomial.

- The normalizability assumption for general singular nonlinear systems is removed. Thus, a complete solution to the output regulation problem of singular nonlinear systems *via* normal output feedback is obtained.

- Some solvability conditions for the robust output regulation problem for nonlinear singular systems are established.

The later two results on output regulation are obtained with local stability. It is interesting to further consider the output regulation problem with global stability. My future work will focus on the following problems:

- Establish a global framework for the output regulation problem that can incorporate more advanced stabilization techniques into the design.

- Further study the global output regulation problem of cascaded systems based on the stabilization result obtained in Chapter 2.

- Pursue some global solutions on the stabilization and output regulation problems of the singular systems studied in Chapters 3 and 4.

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